Online Appendix to Portfolio Choice with Sustainable Spending: A Model of Reaching for Yield

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This online appendix contains proofs of the two propositions as well as additional details on derivations omitted in main text. The order and section numbering follow the main text.

Contents

2 Comparative Statics with Power Utility

2.1 Consumption-Wealth Ratios

Here we derive consumption wealth ratios for standard models portfolio choice models (with no sustainable spending constraint) for CRRA and Epstein-Zin utilities.

CRRA The wealth dynamics is standard

$$\frac{dw_t}{w_t} = (r_f + \alpha\mu)dt + \alpha\sigma dZ_t - \frac{c_t}{w_t}dt$$

The HJB equation is

$$\rho v(w) = \max_{\alpha,c} \left\{ \frac{c^{1-\gamma}}{(1-\gamma)} + v'(w)\left(r_f + \alpha\mu - \frac{c}{w}\right) + \frac{1}{2}v''(w)w^2\alpha^2\sigma^2 \right\}$$

Under the standard guess $v(w) = A\frac{w^{1-\gamma}}{1-\gamma}$ we get

$$\rho A\frac{w^{1-\gamma}}{1-\gamma} = \max_{\alpha,c} \left\{ \frac{c^{1-\gamma}}{(1-\gamma)} + Aw^{1-\gamma}\left(r_f + \alpha\mu - \frac{c}{w}\right) - \frac{1}{2}\gamma A w^{1-\gamma}\alpha^2\sigma^2 \right\}$$

First order condition for α gives the usual

$$\alpha = \frac{\mu}{\gamma\sigma^2}$$

First order condition for consumption gives

$$c^{-\gamma} - Aw^{-\gamma} = 0 \implies \frac{c}{w} = A^{-\frac{1}{\gamma}} \implies c = A^{-\frac{1}{\gamma}}w$$

Now subsitute optimal α and c from above into the maximized HJB to solve for A

$$\rho A\frac{w^{1-\gamma}}{1-\gamma} = \frac{w^{1-\gamma}A^{-\frac{1-\gamma}{\gamma}}}{(1-\gamma)} + Aw^{1-\gamma}\left(r_f + \frac{\mu^2}{\gamma\sigma^2} - A^{-\frac{1}{\gamma}}\right) - \frac{1}{2}Aw^{1-\gamma}\frac{\mu^2}{\gamma\sigma^2}$$

Divide through by $Aw^{1-\gamma}$

$$\frac{\rho}{1-\gamma} = \frac{A^{-\frac{1}{\gamma}}}{1-\gamma} + r_f + \frac{1}{2\gamma}\left(\frac{\mu^2}{\sigma^2}\right) - A^{-\frac{1}{\gamma}}$$

$$\frac{\rho}{1-\gamma} = \frac{\gamma}{1-\gamma}A^{-\frac{1}{\gamma}} + r_f + \frac{1}{2\gamma}\left(\frac{\mu}{\sigma}\right)^2$$

$$\boxed{A^{-\frac{1}{\gamma}} = \frac{c}{w} = \frac{\rho}{\gamma} - \frac{1-\gamma}{\gamma}\left(r_f + \frac{1}{2\gamma}\left(\frac{\mu}{\sigma}\right)^2\right)} \tag{1}$$

Under log utility this gives a standard result that $c = \rho w$.

Using consumption to wealth ratio we can calculate drift of wealth and log wealth under optimal consumption and portfolio allocation

$$\begin{aligned} r_f + \alpha\mu - \frac{c}{w} &= r_f + \frac{\mu^2}{\gamma\sigma^2} - \frac{\rho}{\gamma} + \frac{1-\gamma}{\gamma}\left(r_f + \frac{1}{2\gamma}\left(\frac{\mu}{\sigma}\right)^2\right) \\ &= r_f\left(1 + \frac{1-\gamma}{\gamma}\right) + \frac{1}{\gamma}\left(\frac{\mu}{\sigma}\right)^2\left(1 + \frac{1-\gamma}{2\gamma}\right) - \frac{\rho}{\gamma} \\ &= \frac{r_f - \rho}{\gamma} + \frac{1+\gamma}{2\gamma^2}\left(\frac{\mu}{\sigma}\right)^2 \end{aligned}$$

The drift of log wealth subtracts half variance out of this expression

$$
\begin{aligned}
r_f + \alpha\mu - \frac{c}{w} - \frac{1}{2}\alpha^2\sigma^2 &= \frac{r_f - \rho}{\gamma} + \frac{1+\gamma}{2\gamma^2}\left(\frac{\mu}{\sigma}\right)^2 - \frac{1}{2\gamma^2}\left(\frac{\mu}{\sigma}\right)^2 \\
&= \frac{r_f - \rho}{\gamma} + \frac{1}{2\gamma}\left(\frac{\mu}{\sigma}\right)^2
\end{aligned}
$$

2.2 Proof of Proposition 1

To prove proposition 1 we derive a closed form expression for the risky share α. The relationship between risky share and discount rate is determined by two equations: (1) FOC w.r.t. the risky share and (2) the maximized HJB:

$$
\begin{aligned}
0 &= (r_f + \alpha\mu)^{-\gamma}\mu - B\sigma^2\alpha = 0 \\
0 &= \rho B \frac{w^{1-\gamma}}{1-\gamma} - \frac{(w\,[r_f + \alpha\mu])^{1-\gamma}}{1-\gamma} + \frac{1}{2}B\gamma w^{-\gamma-1} w^2\alpha^2\sigma^2
\end{aligned}
$$

where we substituted our guess for the value function $v(w) = B\frac{w^{1-\gamma}}{1-\gamma}$. Cancel wealth in the second equation to obtain

$$
\begin{aligned}
0 &= (r_f + \alpha\mu)^{-\gamma} \cdot \mu - B\gamma\sigma^2\alpha \\
0 &= \rho B \frac{1}{1-\gamma} - \frac{(r_f + \alpha\mu)^{1-\gamma}}{1-\gamma} + \frac{1}{2}\gamma B\alpha^2\sigma^2
\end{aligned}
$$

The fact that wealth cancels confirms out conjecture for the value function. Use the first equation to express B as a function of α

$$
B = \frac{(r_f + \alpha\mu)^{-\gamma}}{\alpha\gamma\sigma^2}
$$

and substitute it into the second equation. This allows to cancel the term $(r_f + \alpha\mu)^{-\gamma}$:

$$
\frac{\rho\mu}{(1-\gamma)\gamma\sigma^2\alpha} = \frac{r_f + \alpha\mu}{1-\gamma} + \frac{\alpha\mu}{2}
$$

$$
\begin{aligned}
\frac{\rho\mu}{(1-\gamma)\gamma\sigma^2\alpha} &= \alpha\mu\left(\frac{1}{1-\gamma} - \frac{1}{2}\right) + \frac{r_f}{1-\gamma} \\
\alpha^2 \cdot \mu\frac{1+\gamma}{2(1-\gamma)} + \alpha \cdot \frac{r_f}{1-\gamma} - \frac{\rho\mu}{(1-\gamma)\gamma\sigma^2} &= 0 \,\Big|\, \cdot 2(1-\gamma)\gamma\sigma^2 \\
\alpha^2 \cdot \mu(1+\gamma)\gamma\sigma^2 + \alpha \cdot 2r_f\gamma\sigma^2 - 2\rho\mu &= 0
\end{aligned}
$$

The solutions for this equation are

$$\alpha = \frac{-2r_f\gamma\sigma^2 \pm \sqrt{K}}{2\mu(1+\gamma)\gamma\sigma^2},\ K = (2r_f\gamma\sigma^2)^2 + 4\cdot 2\rho\mu\cdot\mu(1+\gamma)\gamma\sigma^2 > 0$$

We further simplify this expression by dividing through by $2\gamma\sigma^2$ and redefining K

$$\alpha = \frac{-r_f \pm \sqrt{K}}{\mu(1+\gamma)}, K = r_f^2 + 2\rho\left(\frac{1+\gamma}{\gamma}\right)\left(\frac{\mu}{\sigma}\right)^2$$

We are interested in the solution where $\alpha > 0$ so that the second order condition is satisfied. Therefore, we take the largest solution.

Effect of Discount Rate We see that K is increasing in ρ so that α will be increasing in ρ

$$\frac{d\alpha}{d\rho} > 0$$

This means that a more impatient investor has a more aggressive asset allocation.

Effect of Risk Free Rate We can also see the effect of risk-free rate on asset allocation using the solution above. This is equivalent to considering the derivative of

$$-r_f + \sqrt{r_f^2 + X}$$

with respect to r_f where X is a constant. Let's take that derivative

$$-1 + \frac{r_f}{\sqrt{r_f^2 + X}}\ vs.\ 0$$

$$\frac{r_f}{\sqrt{r_f^2 + X}}\ vs.\ 1$$

We can see that the left hand side is smaller than one meaning that the derivative of α w.r.t. r_f is negative.

Taking the second derivative we can see that the relationship between α and r_f is convex

$$\frac{\sqrt{r_f^2 + X} - \frac{r_f^2}{\sqrt{r_f^2+X}}}{r_f^2 + X} = \frac{(r_f^2 + X) - r_f^2}{(r_f^2 + X)^{3/2}} > 0$$

Effect of Risk Premium Now we consider the effect of risk premium on the risky share. Use the first order condition and implicit function theorem to write

$$f(\alpha,\mu) \equiv \alpha^2 \cdot \mu(1+\gamma)\gamma\sigma^2 + \alpha \cdot 2r_f\gamma\sigma^2 - 2\rho\mu = 0$$

$$\frac{d\alpha}{d\mu} = -\frac{\partial f/\partial\mu}{\partial f/\partial\alpha} = -\frac{\alpha^2(1+\gamma)\gamma\sigma^2 - 2\rho}{2\alpha\mu(1+\gamma)\gamma\sigma^2 + 2r\gamma\sigma^2}$$

First order condition allows us to sign the numerator:

$$\mu(\alpha^2 \cdot \mu(1+\gamma)\gamma\sigma^2 - 2\rho) = -\alpha \cdot 2r_f\gamma\sigma^2$$

Under $\mu > 0$ we have $\alpha > 0$ and, therefore, the numerator is proportional to $-r_f$. Now lets work with the denominator

$$\begin{aligned}
\frac{\partial f}{\partial\alpha} &= 2\alpha\mu(1+\gamma)\gamma\sigma^2 + 2r_f\gamma\sigma^2 \\
&\propto \alpha\mu(1+\gamma) + r_f \\
[\text{Use solution for } \alpha] &= \frac{-r_f + \sqrt{K}}{\mu(1+\gamma)}\mu(1+\gamma) + r_f \\
&= \sqrt{K} > 0
\end{aligned}$$

Combining both results we get our comparative static

$$\frac{d\alpha}{d\mu} \propto r_f$$

2.3 Proof of Proposition 2

There are two unknowns to be solved: the risky share α and the constant A that reflects investment opportunities. There are jointly determined by the following two equations

$$\left(r_f + \alpha\mu - \frac{1}{2}\alpha^2\sigma^2\right)^{-\gamma}(\mu - \alpha\sigma^2) + (1-\gamma)C\alpha\sigma^2 = 0 \tag{2}$$

and

$$\rho C\frac{1}{1-\gamma} = \frac{(E[d\log V]/dt)^{1-\gamma}}{1-\gamma} + (1-\gamma)\frac{C}{2}\alpha^2\sigma^2. \tag{3}$$

We can use the first equation, the First Order Condition, to express A in terms of other parameters

$$C = \frac{\left(r_f + \alpha\mu - \frac{1}{2}\alpha^2\sigma^2\right)^{-\gamma}(\mu - \alpha\sigma^2)}{(\gamma-1)A\alpha\sigma^2}$$

and substitute into the second equation. This allows us to cancel $(\cdot)^{-\gamma}$ terms. Denote the resulting equation

$$h \equiv \frac{\left(r_f + \alpha\mu - \frac{1}{2}\alpha^2\sigma^2\right)}{1-\gamma} - (1-\gamma)\frac{1}{2}\alpha^2\sigma^2\frac{\mu - \alpha\sigma^2}{(1-\gamma)\alpha\sigma^2} + \rho\frac{(\mu-\alpha\sigma^2)}{(1-\gamma)^2\alpha\sigma^2} = 0 \tag{4}$$

It is not possible to solve this equation for the risky share α in closed form. Nevertheless, we can characterize α using the implicit function theorem. It says that

$$\frac{d\alpha}{d\beta} = -\frac{\partial h/\partial\beta}{\partial h/\partial\alpha}$$

where β is a variable of interest like the risk free rate.

Since all comparative statics depend on $\partial h/\partial\alpha$ we are going to sign it first.

$$\begin{aligned}
\frac{\partial h}{\partial\alpha} &= \frac{\mu}{1-\gamma} - \frac{\alpha\sigma^2}{1-\gamma} - \frac{1}{2}(\mu - \alpha\sigma^2) - \frac{1}{2}\alpha(-\sigma^2) - \frac{\rho\mu}{(1-\gamma)^2\alpha^2\sigma^2} \\
&= \frac{1}{1-\gamma}(\mu - \alpha\sigma^2) - \frac{1}{2}(\mu-\alpha\sigma^2) + \frac{1}{2}\alpha\sigma^2 - \frac{\rho\mu}{(1-\gamma)^2\alpha^2\sigma^2} \\
&= \left(\frac{1}{1-\gamma} - \frac{1}{2}\right)(\mu - \alpha\sigma^2) + \frac{1}{2}\alpha\sigma^2 - \frac{\rho\mu}{(1-\gamma)^2\alpha^2\sigma^2} \\
&= \frac{1+\gamma}{2(1-\gamma)}(\mu - \alpha\sigma^2) + \frac{1}{2}\alpha\sigma^2 - \frac{\rho\mu}{(1-\gamma)^2\alpha^2\sigma^2}
\end{aligned}$$

To proceed go back to equation 4 and note that

$$\underbrace{\frac{\left(r_f + \alpha\mu - \frac{1}{2}\alpha^2\sigma^2\right)}{1-\gamma}}_{<0 \text{ for } \gamma > 1} - \frac{1}{2}\alpha(\mu - \alpha\sigma^2) + \rho\frac{(\mu - \alpha\sigma^2)}{(1-\gamma)^2\alpha\sigma^2} = 0$$

Therefore, we should have

$$-\frac{1}{2}\alpha(\mu - \alpha\sigma^2) + \rho\frac{(\mu - \alpha\sigma^2)}{(1-\gamma)^2\alpha\sigma^2} > 0$$

$$(\mu - \alpha\sigma^2)\left(\frac{\rho}{(1-\gamma)^2\alpha\sigma^2} - \frac{1}{2}\alpha\right) > 0$$

$$\underbrace{(\mu - \alpha\sigma^2)}_{>0 \text{ for } \gamma>1 \text{ from eq. (1)}} \underbrace{\frac{1}{(1-\gamma)\alpha\sigma^2}}_{<0 \text{ for } \gamma>1}\left(\rho\frac{1}{1-\gamma} - \frac{1}{2}(1-\gamma)\alpha^2\sigma^2\right) > 0$$

$$\implies \rho\frac{1}{1-\gamma} - \frac{1}{2}(1-\gamma)\alpha^2\sigma^2 < 0 \implies \frac{1}{2} < \frac{\rho}{(1-\gamma)^2\alpha^2\sigma^2}$$

Multiply both sides by $-\mu$ to finally get

$$-\frac{\rho}{(1-\gamma)^2\alpha^2\sigma^2} < -\frac{1}{2}\mu$$

and use this for out comparative static

$$\begin{aligned}
\frac{\partial h}{\partial \alpha} &< \frac{1+\gamma}{2(1-\gamma)}(\mu - \alpha\sigma^2) + \frac{1}{2}\alpha\sigma^2 - \frac{1}{2}\mu \\
&= \frac{1+\gamma}{2(1-\gamma)}(\mu - \alpha\sigma^2) - \frac{1}{2}(\mu - \alpha\sigma^2) \\
&= \left(\frac{1+\gamma}{2(1-\gamma)} - \frac{1}{2}\right)(\mu - \alpha\sigma^2) \\
&= \underbrace{\frac{\gamma}{1-\gamma}}_{<0}\underbrace{(\mu - \alpha\sigma^2)}_{>0} < 0 \\
\implies \frac{\partial h}{\partial \alpha} &< 0
\end{aligned}$$

Thus we evaluated the denominator of the comparative static and can simplify it to

$$\frac{d\alpha}{d\beta} \propto \frac{\partial h}{\partial \beta}$$

Risk Free Rate From equation (4) $\partial h/\partial r_f < 0$ implying that

$$\frac{d\alpha}{dr_f} < 0$$

The risky share α decreases in the risk free rate r_f.

Convexity in the Risk Free Rate We next prove that $\alpha(r_f)$ is a convex function. To do this differentiate h from equation (4) w.r.t. r_f twice to get

$$0 = h(\alpha(r_f), r_f)$$

$$0 = \left(\frac{\partial^2 h}{\partial \alpha^2}\frac{d\alpha}{dr_f} + \frac{\partial^2 h}{\partial r_f \partial \alpha}\right)\frac{d\alpha}{dr_f} + \frac{\partial h}{\partial \alpha}\frac{d^2\alpha}{dr_f^2} + \frac{\partial^2 h}{\partial r_f \partial \alpha}\frac{d\alpha}{r_f} + \frac{\partial^2 h}{\partial r_f^2}$$

From above we know that $\frac{\partial h}{\partial r_f} = \frac{1}{1-\gamma}$, therefore, $\frac{\partial^2 h}{\partial r_f \partial \alpha} = \frac{\partial^2 h}{\partial \alpha \partial r_f} = \frac{\partial^2 h}{\partial r_f^2} = 0$. We get

$$0 = \frac{\partial^2 h}{\partial \alpha^2}\left(\frac{d\alpha}{dr_f}\right)^2 + \frac{\partial h}{\partial \alpha}\frac{d^2\alpha}{dr_f^2}$$

$$\frac{d^2\alpha}{dr_f^2} = -\frac{\frac{\partial^2 h}{\partial \alpha^2}\left(\frac{d\alpha}{dr_f}\right)^2}{\frac{\partial h}{\partial \alpha}}$$

We already signed $\frac{\partial h}{\partial \alpha} < 0$:

$$\frac{d^2\alpha}{dr_f^2} \propto \frac{\partial^2 h}{\partial \alpha^2} = -\frac{1+\gamma}{2(1-\gamma)}\sigma^2 + \frac{\sigma^2}{2} + 2\frac{\rho\mu}{(1-\gamma)^2\alpha^3\sigma^2} > 0$$

since $\gamma > 1$ and $\alpha > 0$.

Discount Rate From equation (4) $\partial h/\partial \rho < 0$ implying that

$$\frac{d\alpha}{d\rho} > 0$$

The risky share α increases in the discount rate ρ.

Risk Premium First we find a value of the risk free rate $\tilde{r}_f$ such that the risky share doesn't depends on the risk premium. Using equation (4) once again and collecting the terms with μ

$$h(\alpha, \mu) = \frac{r_f}{1-\gamma} + \mu\left(\frac{\alpha}{1-\gamma} - \frac{\alpha}{2} + \frac{\rho}{(1-\gamma)^2\alpha\sigma^2}\right) - \frac{1}{2}\frac{\alpha^2\sigma^2}{1-\gamma} + \frac{1}{2}\alpha^2\sigma^2 - \frac{\rho}{(1-\gamma)^2} = 0$$

If the optimal α doesn't depend on μ then the expression multiplying it should be equal to zero. This gives us the the risk share

$$\frac{\alpha}{1-\gamma} - \frac{\alpha}{2} + \frac{\rho}{(1-\gamma)^2\alpha\sigma^2} = 0$$

$$\alpha^2 \frac{1+\gamma}{2(1-\gamma)} + \frac{\rho}{(1-\gamma)^2\sigma^2} = 0$$

$$\alpha = \tilde{\alpha} \equiv \sqrt{\frac{2\rho}{(\gamma^2-1)\sigma^2}}$$

where we pick a positive solution. Substituting this into the FOC we can derive the expression for the risk free rate that makes α indifferent to μ:

$$\frac{r_f}{1-\gamma} - \frac{1}{2}\frac{\alpha^2\sigma^2}{1-\gamma} + \frac{1}{2}\alpha^2\sigma^2 - \frac{\rho}{(1-\gamma)^2} = 0$$

$$r_f - \frac{1}{2}\alpha^2\sigma^2 + \frac{1}{2}\alpha^2\sigma^2(1-\gamma) - \frac{\rho}{1-\gamma} = 0$$

$$r_f - \gamma\frac{1}{2}\alpha^2\sigma^2 + \frac{\rho}{\gamma-1} = 0$$

$$r_f - \gamma\frac{\rho}{(\gamma^2-1)} + \frac{\rho}{\gamma-1} = 0$$

$$r_f = \frac{\gamma\rho}{(\gamma^2-1)} - \frac{\rho}{\gamma-1} = 0$$

$$r_f = \frac{\gamma\rho - \rho(\gamma+1)}{(\gamma^2-1)}$$

$$r_f = \tilde{r}_f \equiv -\frac{\rho}{\gamma^2-1} < 0$$

Note that for $\gamma > 1$ $\tilde{r} < 0$

We then verify that this is the condition the determines whether α increases or decreases with μ. Using the implicit function theorem we have

$$\frac{d\alpha}{d\mu} = -\frac{\partial h/\partial\mu}{\partial h/\partial\alpha}$$

From previous derivations we know that $\partial h/\partial\alpha < 0$. Therefore

$$\begin{aligned}
\frac{d\alpha}{d\mu} \propto \frac{\partial h}{\partial\mu} &= \frac{\alpha}{1-\gamma} - \frac{\alpha}{2} + \frac{\rho}{(1-\gamma)^2\alpha\sigma^2} \\
&= \alpha\frac{1+\gamma}{2(1-\gamma)} + \frac{\rho}{(1-\gamma)^2\alpha\sigma^2} \\
&\propto -\alpha^2\frac{1+\gamma}{2(\gamma-1)} + \frac{\rho}{(\gamma-1)^2\sigma^2} = \begin{cases} \geq 0 \text{ for } \alpha \in [-\tilde{\alpha}, \tilde{\alpha}] \\ < 0 \text{ otherwise} \end{cases}
\end{aligned}$$

Where $\tilde{\alpha}$ is defined above. Since α decreases in r_f, $\alpha < \tilde{\alpha}$ implied $r_f > \tilde{r}_f$ defined above. In this region $\frac{\partial h}{\partial \mu} > 0 \implies \frac{d\alpha}{d\mu} > 0$ – optimal risky share increases in the risk premium. When $\alpha > \tilde{\alpha}$ which happens when $r_f < \tilde{r}_f$ we have that $\frac{\partial h}{\partial \mu} < 0 \implies \frac{d\alpha}{d\mu} < 0$ – optimal risky share decreases in the risk premium.

3 Extensions of the Static Model

3.1 Gifts

Arithmetic Average Model In the presence of gifts the budget constraint and the arithmetic consumption rule become

$$dw_t = w_t dr_{p,t} + w_t(g_u + g_e) - c_t dt$$
$$c_t dt = w_t(E_t dr_{p,t} + g_u) = w_t(r_f + g_u + \alpha\mu)dt$$

Substitute consumption rule into the budget constraint to ge

$$\begin{aligned} dw_t &= w_t(r_f + \alpha\mu) + w_t\alpha\sigma dZ_t + w_t(g_u + g_e) - w_t(r_f + g_u + \alpha\mu)dt \\ &= w_t g_e dt + w_t\alpha\sigma dZ_t \end{aligned}$$

The process for log consumption coincides with the process for log wealth

$$d\log(w_t) = d\log(c_t) = \left(g_e - \frac{1}{2}\alpha^2\sigma^2\right) dt + \alpha\sigma dZ_t$$

Such that the portfolio constraint and the iso-value curves from the mean standard deviation analysis can be written as

$$c_0 = r_f + g_u + \frac{\mu}{\sigma}\sigma_c$$
$$c_0 = \left[\left(\rho + (\gamma-1)g_e - \gamma(\gamma-1)\frac{\sigma_c^2}{2}\right)(1-\gamma)v\right]^{\frac{1}{1-\gamma}}$$

We see that current use gifts act as increasing the risk free rate and therefore reduce risk taking. On the other, endowment gifts act as increasing the discount rate and therefore increase risk taking.

We now show present the same argument by writing down the HJB equation under the usual guess and showing the equivalence

$$\rho A\frac{w^{1-\gamma}}{1-\gamma} = \max_\alpha \left\{ \frac{\left(w\left(r_f + g_u + \alpha\mu\right)\right)^{1-\gamma}}{1-\gamma} + Aw^{1-\gamma}g_e - \frac{\gamma}{2}Aw^{1-\gamma}\alpha^2\sigma^2 \right\}$$

The term $Aw^{1-\gamma}g_e$ doesn't depend on α. Thus we can move it outside of the maximization and bring it on the other side

$$\left(\rho + (\gamma-1)g_e\right)A\frac{w^{1-\gamma}}{1-\gamma} = \max_\alpha \left\{ \frac{\left(w\left(r_f + g_u + \alpha\mu\right)\right)^{1-\gamma}}{1-\gamma} - \frac{\gamma}{2}Aw^{1-\gamma}\alpha^2\sigma^2 \right\}$$

We arrive to the same conclusion as with the mean standard deviation analysis

Geometric Average Model In the presence of gifts the budget constraint and the arithmetic consumption rule become

$$dw_t = w_t dr_{p,t} + w_t(g_u + g_e) - c_t dt$$

$$c_t dt = w_t \left(E_t dr_{p,t} + g_u - \frac{1}{2}\alpha^2\sigma^2 dt \right) = w_t \left(r_f + g_u + \alpha\mu - \frac{1}{2}\alpha^2\sigma^2 \right) dt$$

Substitute consumption rule into the budget constraint to ge

$$\begin{aligned} dw_t &= w_t(r_f + \alpha\mu) + w_t\alpha\sigma dZ_t + w_t(g_u + g_e) - w_t \left(r_f + g_u + \alpha\mu - \frac{1}{2}\alpha^2\sigma^2 \right) dt \\ &= w_t \left(g_e + \frac{1}{2}\alpha^2\sigma^2 \right) dt + w_t\alpha\sigma dZ_t \end{aligned}$$

The process for log consumption coincides with the process for log wealth

$$d\log(w_t) = d\log(c_t) = g_e dt + \alpha\sigma dZ_t$$

Such that the portfolio constraint and the iso-value curves from the mean standard deviation analysis can be written as

$$c_0 = r_f + g_u + \frac{\mu}{\sigma}\sigma_c$$

$$c_0 = \left[\left(\rho + (\gamma - 1)g_e - (\gamma - 1)^2 \frac{\sigma_c^2}{2} \right) (1 - \gamma)v \right]^{\frac{1}{1-\gamma}}$$

The effect of gifts is exactly the same as in the arithmetic average model. As with the arithmetic average model we can show the same point using an HJB equation.

3.2 Inflation and Nominal Spending Rules

Consider a price level p_t following $dp_t = p_t\pi dt$ where π is inflation rate. The nominal rate becomes $r_f^\$ = r_f + \pi$ and the nominal return on the risky asset $dr_t^\$ = (r_f + \pi + \mu)dt + \sigma dZ_t$.

Arithmetic Average Model Suppose that the investor has a nominal sustainable spending constraint

$$c_t^\$ dt = w_t^\$ E[dr_{p,t}^\$]$$

where $c_t^\$ = c_t p_t$ and $w_t^\$ = w_t p_t$ so that

$$c_t dt = w_t E[dr_{p,t}^\$] = w_t(r_f^\$ + \alpha\mu)dt$$

The law of motion for nominal wealth is then

$$\begin{aligned}\frac{dw_t^{\$}}{w_t^{\$}} &= \alpha dr_t^{\$} + (1-\alpha) r_f^{\$} dt - \frac{c_t^{\$}}{w_t^{\$}} dt \\ &= \alpha(r_f^{\$} + \mu) dt + \alpha\sigma dZ_t + (1-\alpha) r_f^{\$} dt - E[dr_{p,t}^{\$}] \\ &= \alpha\sigma dZ_t\end{aligned}$$

This implies that real wealth follows

$$\frac{dw_t}{w_t} = \frac{dw_t^{\$}}{w_t^{\$}} - \pi dt = -\pi dt + \alpha\sigma dZ_t$$

Log consumption then follows

$$\begin{aligned}d\log(c_t) &= d\log(w_t) + d\log(E[dr_{p,t}^{\$}]) \\ &= \underbrace{\left(-\pi - \frac{\alpha^2\sigma^2}{2}\right)}_{\mu_c} dt + \underbrace{\alpha\sigma}_{\sigma_c} dZ_t\end{aligned}$$

We can now rewrite the portfolio constraint and iso-value curves as

$$\begin{aligned}c_0 &= r_f + \pi + \frac{\mu}{\sigma}\sigma_c \\ c_0 &= \left[\left(\rho - (\gamma-1)\pi - \gamma(\gamma-1)\frac{\sigma_c^2}{2}\right)(1-\gamma)v\right]^{\frac{1}{1-\gamma}}\end{aligned}$$

This shows that a nominal spending rule with positive inflation acts as a higher risk free rate **and** a lower discount rate. Both reduce risk taking so that inflation also reduces risk taking. As we did in the analysis of gifts we can arrive to the same conclusion by writing the HJB.

Geometric Average Model Now the spending rule is

$$c_t^{\$} dt = w_t^{\$} E[d\log V_t^{\$}]$$

where $V_t^{\$}$ is defined as the solution to

$$\frac{dV_t^{\$}}{V_t^{\$}} = (r_f^{\$} + \alpha\mu) dt + \alpha\sigma dZ_t$$

so that

$$c_t^{\$} dt = w_t^{\$} \left(r_f^{\$} + \alpha\mu - \frac{1}{2}\alpha^2\sigma^2\right) dt$$

The law of motion for nominal wealth is

$$
\begin{aligned}
\frac{dw_t^{\$}}{w_t^{\$}} &= \alpha dr_t^{\$} + (1-\alpha) r_f^{\$} dt - \frac{c_t^{\$}}{w_t^{\$}} dt \\
&= (r_f^{\$} + \mu) dt + \alpha\sigma dZ_t + (1-\alpha) r_f^{\$} dt - \left(r_f^{\$} + \alpha\mu - \frac{1}{2}\alpha^2\sigma^2 \right) dt \\
&= \frac{1}{2}\alpha^2\sigma^2 dt + \alpha\sigma dZ_t
\end{aligned}
$$

This implies the following process for log consumption

$$
\begin{aligned}
d\log(c_t) &= d\log(w_t) \\
&= \underbrace{-\pi}_{\mu_c} dt + \underbrace{\alpha\sigma}_{\sigma_c} dZ_t
\end{aligned}
$$

We can now rewrite the portfolio constraint and iso-value curves as

$$
\begin{aligned}
c_0 &= r_f + \pi + \frac{\mu}{\sigma}\sigma_c - \frac{1}{2}\alpha^2\sigma^2 \\
c_0 &= \left[\left(\rho - (\gamma-1)\pi - (\gamma-1)^2 \frac{\sigma_c^2}{2} \right) (1-\gamma) v \right]^{\frac{1}{1-\gamma}}
\end{aligned}
$$

Inflation enters in the same way as it did for the arithmetic average model. As we did in the analysis of gifts we can arrive to the same conclusion by writing the HJB.

3.3 Epstein-Zin Preferences

Here we show how to extend the geometric average model to stochastic differential utility. We show that all the results of the power utility carry through regardless of the value of EIS ψ provided risk aversion$\gamma > 1$. Lifetime utility over such preferences is defined as the solution to the following equation

$$V_t = E_t \int_t^T f(c_s, V_s) ds$$

We have the same sustainable spending constraint $c_t = w_t \left(r_f + \alpha\mu + \frac{1}{2}\alpha^2\sigma^2 \right)$ implying law of motion for wealth

$$dw_t = w_t \frac{1}{2}\alpha^2\sigma^2 dt + w_t \alpha\sigma dZ_t$$

The HJB equation is

$$0 = \max_{\alpha} f(c, V) + \frac{\partial V}{\partial W} \cdot w_t \frac{1}{2}\alpha^2\sigma^2 + \frac{1}{2}\frac{\partial^2 V}{\partial W^2} \cdot w^2\alpha^2\sigma^2 \tag{5}$$

Epstein-Zin form is defined by

$$f(c, V) = \frac{1}{1-\psi^{-1}} \left[\frac{\rho c^{1-\psi^{-1}}}{((1-\gamma)V)^{\frac{\gamma-\psi^{-1}}{1-\gamma}}} - \rho(1-\gamma)V \right] \tag{6}$$

3.3.1 Arithmetic Average Model

Here we explicitly solve for the risk share in the arithmetic average case. The solution approach will be exactly the same as in the case of power utility from before. First, we conjecture a value function $V(w) = A\frac{w^{1-\gamma}}{1-\gamma}$. Second, use the FOC to express A as a function of all the other variables. Third, substitute A into the HJB and solve for α.

After substituting consumption $c = w(r_f + \alpha\mu)$ (implying that drift of wealth is zero) into the HJB we get

$$0 = \max_{\alpha} \left\{ \frac{1}{1-\psi^{-1}} \left[\frac{\rho(w(r_f + \alpha\mu))^{1-\psi^{-1}}}{(Aw^{1-\gamma})^{\frac{\gamma-\psi^{-1}}{1-\gamma}}} - \rho A w^{1-\gamma} \right] - \frac{1}{2}\gamma A w^{1-\gamma}\alpha^2\sigma^2 \right\}$$

we can factor $Aw^{1-\gamma}$ out of the maximization problem

$$0 = Aw^{1-\gamma} \max_{\alpha} \left\{ \frac{1}{1-\psi^{-1}} \left[\frac{\rho(r_f + \alpha\mu)^{1-\psi^{-1}}}{A^{\frac{1-\psi^{-1}}{1-\gamma}}} - \rho \right] - \gamma\frac{1}{2}\alpha^2\sigma^2 \right\}$$

Use first order condition to express $1/A^{\frac{1-\psi^{-1}}{1-\gamma}}$

$$\rho\frac{(r_f+\alpha\mu)^{-\psi^{-1}}\mu}{A^{\frac{1-\psi^{-1}}{1-\gamma}}}-\gamma\alpha\sigma^2=0 \implies \frac{1}{A^{\frac{1-\psi^{-1}}{1-\gamma}}}=\frac{\gamma\alpha\sigma^2}{\rho(r_f+\alpha\mu)^{-\psi^{-1}}\mu}$$

and substitute it back into the maximized HJB

$$0=\frac{1}{1-\psi^{-1}}\left[\frac{(r_f+\alpha\mu)\gamma\alpha\sigma^2}{\mu}-\rho\right]-\gamma\frac{1}{2}\alpha^2\sigma^2$$

Rearrange to get a standard quadratic equation:

$$\gamma\sigma^2\frac{1+\psi^{-1}}{2(1-\psi^{-1})}\alpha^2+\frac{\gamma\sigma^2 r_f}{(1-\psi^{-1})\mu}\alpha-\frac{\rho}{1-\psi^{-1}}=0\ \Bigg|\cdot\frac{(1-\psi^{-1})\mu}{\gamma\sigma^2}$$

$$\frac{\mu(1+\psi^{-1})}{2}\alpha^2+r_f\alpha-\frac{\rho\mu}{\gamma\sigma^2}=0$$

The positive solution to this equation is

$$\alpha=\frac{-r_f+\sqrt{L}}{\mu(1+\psi^{-1})},\ L=r_f^2+2\rho\frac{1+\psi^{-1}}{\gamma}\left(\frac{\mu}{\sigma}\right)^2$$

This solution coincides with the solution from CRRA utility when $\gamma=\psi^{-1}$.

3.3.2 Geometric Average Model

Now we proceed to deriving the first order condition and comparative statics. We make the same guess that $V(w)=\frac{w^{1-\gamma}}{1-\gamma}$ where A is an unknown constant. Substitute the consumption constraint $c_t=w_t\left(r_f+\alpha\mu+\frac{1}{2}\alpha^2\sigma^2\right)$, $f(c,V)$ from above, the guess for the value function into equation (5) and factor out $w^{1-\gamma}$ and A to get

$$0=Aw^{1-\gamma}\max_{\alpha}\left\{\frac{1}{1-\psi^{-1}}\left[\frac{\rho\left(r_f+\alpha\mu-\frac{1}{2}\alpha^2\sigma^2\right)^{1-\psi^{-1}}}{A^{\frac{1-\psi^{-1}}{1-\gamma}}}-\rho\right]+\frac{1}{2}(1-\gamma)\alpha^2\sigma^2\right\}$$

Before going further consider a limiting case when $\psi\to 0\Rightarrow\psi^{-1}\to\infty$. Then

$$\frac{1}{1-\psi^{-1}}\to 0,\left(r_f+\alpha\mu-\frac{1}{2}\alpha^2\sigma^2\right)^{1-\psi^{-1}}\to 0, A^{\frac{1-\psi^{-1}}{1-\gamma}}\to\infty$$

Therefore, the whole first term goes to zero leaving us with

$$0=Aw^{1-\gamma}\max_{\alpha}\left\{-\frac{1}{2}(\gamma-1)\alpha^2\sigma^2\right\}$$

which result in the optimal portfolio rule $\alpha = 0$. Notice, however, that since we require consumption to stay positive the limiting case will only have a solution $\alpha = 0$ when $r_f > 0$.

Now return to the HJB equation. The first order condition is

$$\rho\left(r_f + \alpha\mu - \frac{1}{2}\alpha^2\sigma^2\right)^{-\psi^{-1}}(\mu - \alpha\sigma^2) + A^{\frac{1-\psi^{-1}}{1-\gamma}}(1-\gamma)\alpha\sigma^2 = 0 \tag{7}$$

This allows to express $A^{\frac{1-\psi^{-1}}{1-\gamma}}$ and substitute it into the HJB equation to get

$$\frac{1}{1-\psi^{-1}}\left[\frac{\left(r_f + \alpha\mu - \frac{1}{2}\alpha^2\sigma^2\right)(\gamma-1)\alpha\sigma^2}{\mu - \alpha\sigma^2} - \rho\right] - \frac{1}{2}(\gamma-1)\alpha^2\sigma^2 = 0$$

We rearrange it to get the expression that we will work with

$$h(\alpha, r_f, \rho, \gamma, \psi) \equiv \left(r_f + \alpha\mu - \frac{1}{2}\alpha^2\sigma^2\right) - \rho\frac{\mu - \alpha\sigma^2}{(\gamma-1)\alpha\sigma^2} - \frac{1}{2}\alpha(\mu - \alpha\sigma^2)(1-\psi^{-1}) = 0 \tag{8}$$

First, we can see from equation (8) that

$$\rho\frac{\mu - \alpha\sigma^2}{(\gamma-1)\alpha\sigma^2} + \frac{1}{2}\alpha(\mu - \alpha\sigma^2)(1-\psi^{-1}) > 0$$

$$\rho > -\frac{1}{2}\alpha^2\sigma^2(1-\psi^{-1})(\gamma-1) \tag{9}$$

which is trivial when both $\gamma > 1$ and $\psi > 1$ but we will use this condition for a more general case when $\gamma > 1$. Implicit function theorem implies that

$$\frac{d\alpha}{d\beta} = -\frac{\partial h/\partial\beta}{\partial h/\partial\alpha}$$

where β is any variable that we are interested in. We now sign $\frac{\partial h}{\partial \alpha}$

$$\begin{aligned}
\frac{\partial h}{\partial \alpha} &= (\mu - \alpha\sigma^2) - \rho\frac{-\sigma^2(\gamma-1)\alpha\sigma^2 - (\gamma-1)\sigma^2(\mu - \alpha\sigma^2)}{[(\gamma-1)\alpha\sigma^2]^2} - \frac{1}{2}(1-\psi^{-1})(\mu - 2\alpha\sigma^2) \\
&= (\mu - \alpha\sigma^2) + \frac{\rho\mu}{(\gamma-1)\alpha^2\sigma^2} - \frac{1}{2}(1-\psi^{-1})(\mu - \alpha\sigma^2) + \frac{1}{2}(1-\psi^{-1})\alpha\sigma^2 \\
&= \left(1 - \frac{1}{2}(1-\psi^{-1})\right)(\mu - \alpha\sigma^2) + \frac{\rho\mu}{(\gamma-1)\alpha^2\sigma^2} + \frac{1}{2}(1-\psi^{-1})\alpha\sigma^2 \\
&= \frac{1}{2}(1+\psi^{-1})(\mu - \alpha\sigma^2) + \frac{\mu}{(\gamma-1)\alpha^2\sigma^2}\rho + \frac{1}{2}(1-\psi^{-1})\alpha\sigma^2
\end{aligned}$$

Now use the inequality (9)

$$\begin{aligned}
\frac{\partial h}{\partial \alpha} &> \frac{1}{2}(1+\psi^{-1})(\mu-\alpha\sigma^2) - \frac{\mu}{(\gamma-1)\alpha^2\sigma^2}\frac{1}{2}\alpha^2\sigma^2(1-\psi^{-1})(\gamma-1) + \frac{1}{2}(1-\psi^{-1})\alpha\sigma^2 \\
&= \frac{1}{2}(1+\psi^{-1})(\mu-\alpha\sigma^2) - \mu\frac{1}{2}(1-\psi^{-1}) + \frac{1}{2}(1-\psi^{-1})\alpha\sigma^2 \\
&= \frac{1}{2}(1+\psi^{-1})(\mu-\alpha\sigma^2) - \frac{1}{2}(\mu-\alpha\sigma^2)(1-\psi^{-1}) \\
&= \psi^{-1}(\mu-\alpha\sigma^2) > 0
\end{aligned}$$

Therefore, we have

$$\frac{\partial h}{\partial \alpha} > 0 \implies \frac{d\alpha}{d\beta} \propto -\frac{\partial h}{\partial \beta}$$

Risk Free Rate Use the result derived above to sign the effect of the risk free rate on the risky share

$$\frac{d\alpha}{dr_f} \propto -\frac{\partial h}{\partial r_f} = -1 < 0$$

Higher risk free rate leads to lower risky share

Discount Rate The effect of the discount rate on thr risky share

$$\frac{d\alpha}{d\rho} \propto -\frac{\partial h}{\partial \rho} = \frac{\mu-\alpha\sigma^2}{(\gamma-1)\alpha\sigma^2} > 0$$

Higher discount rate leads to a higher risky share

Risk Premium Similar to power utility the sign of the effect of risk premium on the risky share depends on the risk free rate. First, collect the terms with μ in equation (8)

$$h = \left(r_f - \frac{1}{2}\alpha^2\sigma^2\right) + \frac{\rho}{\gamma-1} + \frac{1}{2}\alpha^2\sigma^2(1-\psi^{-1}) + \mu\left[\alpha - \frac{\rho}{(\gamma-1)\alpha\sigma^2} - \frac{1}{2}\alpha(1-\psi^{-1})\right] = 0$$

Risky share α doesn't change with the risk premium μ when $\alpha = \tilde{\alpha}$ suc that the expression in the brackets is exactly zero

$$\tilde{\alpha} - \frac{\rho}{(\gamma-1)\tilde{\alpha}\sigma^2} - \frac{1}{2}\tilde{\alpha}(1-\psi^{-1}) = 0$$

$$\tilde{\alpha}^2 = \frac{2\rho}{(\gamma-1)(\psi^{-1}+1)\sigma^2}$$

To find the risk free rate that leads to $\alpha = \tilde{\alpha}$ express r_f from h and substitute $\tilde{\alpha}^2$ (so that the terms in the brackets cancel)

$$\begin{aligned}
\tilde{r}_f &= \frac{\psi^{-1}}{2}\tilde{\alpha}^2\sigma^2 - \frac{\rho}{\gamma-1} \\
&= \frac{\psi^{-1}}{2}\frac{2\rho}{(\gamma-1)(\psi^{-1}+1)\sigma^2}\sigma^2 - \frac{\rho}{\gamma-1} \\
&= \frac{\psi^{-1}\rho}{(\gamma-1)(\psi^{-1}+1)} - \frac{\rho}{\gamma-1} \\
&= \frac{\rho}{\gamma-1}\left(\frac{\psi^{-1}}{(\psi^{-1}+1)} - 1\right) \\
&= -\frac{\rho}{(\gamma-1)(\psi^{-1}+1)} < 0
\end{aligned}$$

Hence, for $r_f < \tilde{r}_f$ the effect of risk premium on the risky share is negative and for $r_f > \tilde{r}_f$ the effect of risk premium on the risky share is positive.

Elasticity of Intertemporal Substitution The effect of EIS ψ on the risky share is

$$\frac{d\alpha}{d\psi} \propto -\frac{\partial h}{\partial \psi} = \frac{1}{2}\alpha(\mu - \alpha\sigma^2)\psi^{-2} > 0$$

Higher Elasticity of Intertemporal Substitution leads to a larger risky share.

3.3.3 Consumption-Wealth Ratio

Here we derive the consumption to wealth ratio The HJB equation under the same wealth process is

$$0 = \max_{\alpha,c}\left\{\frac{1}{1-\psi^{-1}}\left[\frac{\rho c^{1-\psi^{-1}}}{((1-\gamma)v(w))^{\frac{\gamma-\psi^{-1}}{1-\gamma}}} - \rho(1-\gamma)v(w)\right] + v'(w)w\left(r_f + \alpha\mu - \frac{c}{w}\right) + \frac{1}{2}v''(w)w^2\alpha^2\sigma^2\right\}$$

The guess for the value function is also $v(w) = A\frac{w^{1-\gamma}}{1-\gamma}$

$$0 = \max_{\alpha,c}\left\{\frac{1}{1-\psi^{-1}}\left[\frac{\rho c^{1-\psi^{-1}}}{(Aw^{1-\gamma})^{\frac{\gamma-\psi^{-1}}{1-\gamma}}} - \rho A w^{1-\gamma}\right] + Aw^{1-\gamma}\left(r_f + \alpha\mu - \frac{c}{w}\right) - \frac{1}{2}\gamma A w^{1-\gamma}\alpha^2\sigma^2\right\}$$

First order condition for α gives the usual

$$\alpha = \frac{\mu}{\gamma\sigma^2}$$

First order condition for c gives

$$\frac{\rho c^{-\psi^{-1}}}{(A)^{\frac{\gamma-\psi^{-1}}{1-\gamma}} w^{\gamma-\psi^{-1}}} - Aw^{-\gamma} = 0 \implies \left(\frac{c}{w}\right)^{-\psi^{-1}} = \frac{1}{\rho} A^{\frac{1-\psi^{-1}}{1-\gamma}} \implies \frac{c}{w} = \rho^{\psi} A^{\frac{1-\psi}{1-\gamma}} \implies c = \rho^{\psi} A^{\frac{1-\psi}{1-\gamma}} w$$

Substitute optimal c and α into the maximized HJB

$$0 = \frac{1}{1-\psi^{-1}} \left[\frac{\rho w^{1-\psi^{-1}} \rho^{\psi(1-\psi^{-1})} A^{\frac{(1-\psi)(1-\psi^{-1})}{1-\gamma}}}{A^{\frac{\gamma-\psi^{-1}}{1-\gamma}} w^{\gamma-\psi^{-1}}} - \rho A w^{1-\gamma} \right] + Aw^{1-\gamma} \left(r_f + \frac{1}{\gamma} \left(\frac{\mu}{\sigma}\right)^2 - \rho^{\psi} A^{\frac{1-\psi}{1-\gamma}} \right) - \frac{1}{2} A w^{1-\gamma} \frac{1}{\gamma} \left(\frac{\mu}{\sigma}\right)^2$$

Divide through by $Aw^{1-\gamma}$

$$0 = \frac{1}{1-\psi^{-1}} \left[\rho^{\psi} A^{\frac{1-\psi}{1-\gamma}} - \rho \right] + r_f + \frac{1}{2\gamma} \left(\frac{\mu}{\sigma}\right)^2 - \rho^{\psi} A^{\frac{1-\psi}{1-\gamma}}$$

$$\frac{\psi^{-1}}{1-\psi^{-1}} \rho^{\psi} A^{\frac{1-\psi}{1-\gamma}} = \frac{\rho}{1-\psi^{-1}} - \left(r_f + \frac{1}{2\gamma} \left(\frac{\mu}{\sigma}\right)^2 \right)$$

$$\boxed{\rho^{\psi} A^{\frac{1-\psi}{1-\gamma}} = \frac{c}{w} = \frac{\rho}{\psi^{-1}} - \frac{1-\psi^{-1}}{\psi^{-1}} \left(r_f + \frac{1}{2\gamma} \left(\frac{\mu}{\sigma}\right)^2 \right)} \tag{10}$$

which coincides with consumption to wealth ratio for CRRA when $\gamma = \psi^{-1}$ ensuring that the derivation is correct.

Using consumption to wealth ratio we can calculate drift of wealth

$$\begin{aligned}
r_f + \alpha\mu - \frac{c}{w} &= r_f + \frac{1}{\gamma} \left(\frac{\mu}{\sigma}\right)^2 - \frac{\rho}{\psi^{-1}} + \frac{1-\psi^{-1}}{\psi^{-1}} \left(r_f + \frac{1}{2\gamma} \left(\frac{\mu}{\sigma}\right)^2 \right) \\
&= r_f \left(1 + \frac{1-\psi^{-1}}{\psi^{-1}} \right) + \frac{1}{\gamma} \left(\frac{\mu}{\sigma}\right)^2 \left(1 + \frac{1-\psi^{-1}}{2\psi^{-1}} \right) - \frac{\rho}{\psi^{-1}} \\
&= \frac{r_f - \rho}{\psi^{-1}} + \frac{1+\psi^{-1}}{2\gamma\psi^{-1}} \left(\frac{\mu}{\sigma}\right)^2
\end{aligned}$$

Drift of log wealth subtracts half variance from that expression

$$\begin{aligned}
r_f + \alpha\mu - \frac{c}{w} - \frac{1}{2}\alpha^2\sigma^2 &= \frac{r_f - \rho}{\psi^{-1}} + \frac{1+\psi^{-1}}{2\gamma\psi^{-1}} \left(\frac{\mu}{\sigma}\right)^2 - \frac{1}{2}\frac{1}{\gamma^2} \left(\frac{\mu}{\sigma}\right)^2 \\
&= \frac{r_f - \rho}{\psi^{-1}} + \frac{1}{2\gamma} \left(\frac{\mu}{\sigma}\right)^2 \left(\frac{1+\psi^{-1}}{\psi^{-1}} - \frac{1}{\gamma} \right)
\end{aligned}$$

In particular, suppose that $\psi \to 0 \Rightarrow \psi^{-1} \to \infty$. Then drift of log wealth and, as a result of log consumption ($d\log(c) = d\log\left(\frac{c}{w}w\right) = d\log\left(\frac{c}{w}w\right) + d\log(w) = d\log(w)$) is

$$\frac{1}{2\gamma}\left(\frac{\mu}{\sigma}\right)^2\left(1-\frac{1}{\gamma}\right)$$

which is positive when $\gamma > 1$ and is equal to zero only when $\gamma = 1$

Understanding limiting case via a continuous time limit Consider a discrete time EZ where we take a continuous limit by considering time increments of length dt

$$V_t = \left[(1-e^{-\rho dt})c_t^{1-\psi^{-1}} + e^{-\rho dt}E_t\left[V_{t+dt}^{1-\gamma}\right]^{\frac{1-\psi^{-1}}{1-\gamma}}\right]^{\frac{1}{1-\psi^{-1}}}$$

A limit as $\psi \to 0 \Rightarrow \psi^{-1} \to -\infty$ is a standard result (e.g. Arrow et al. (1961)) and gives Leontief preferences

$$V_t = \min\left\{c_t, E_t\left[V_{t+dt}^{1-\gamma}\right]^{\frac{1}{1-\gamma}}\right\}$$

Optimality requires

$$c_t^{1-\gamma} = E_t\left[V_{t+dt}^{1-\gamma}\right]$$

Define $U_t = V_t^{1-\gamma}$, guess the functional form $U = Aw^{1-\gamma}$ and do a second order Taylor expansion

$$\begin{aligned}
c_t^{1-\gamma} &= U_t + E_t\left[dU_t\right] \\
&= Aw^{1-\gamma} + E_t\left[(1-\gamma)Aw^{-\gamma}dw\right] - E_t\left[\gamma(1-\gamma)\frac{1}{2}Aw^{-\gamma-1}(dw)^2\right] \\
&= Aw^{1-\gamma} + E_t\left[(1-\gamma)Aw^{-\gamma}w\left(r_f + \alpha\mu - \frac{c}{w}\right)dt + w\alpha\sigma dZ_t\right] - E_t\left[\gamma(1-\gamma)\frac{1}{2}Aw^{-\gamma-1}w^2\alpha^2\sigma^2 dt\right] \\
&= Aw^{1-\gamma} + (1-\gamma)Aw^{1-\gamma}\left(r_f + \alpha\mu - \frac{c}{w}\right)dt - \gamma(1-\gamma)\frac{1}{2}Aw^{1-\gamma}\alpha^2\sigma^2 dt
\end{aligned}$$

Substituting this c_t (which is equal to the whole expression above raised to the power $1-\gamma$) back into the value function

$$V_t = \min\left\{c_t, E_t\left[V_{t+dt}^{1-\gamma}\right]^{\frac{1}{1-\gamma}}\right\} \Rightarrow U^{\frac{1}{1-\gamma}} = (c_t^{1-\gamma})^{\frac{1}{1-\gamma}}$$

$$\Rightarrow \left(Aw^{1-\gamma}\right)^{\frac{1}{1-\gamma}} = \left[Aw^{1-\gamma} + (1-\gamma)Aw^{1-\gamma}\left(r_f + \alpha\mu - \frac{c}{w}\right)dt - \gamma(1-\gamma)\frac{1}{2}Aw^{1-\gamma}\alpha^2\sigma^2 dt\right]^{\frac{1}{1-\gamma}}$$

$$\Rightarrow (1-\gamma)Aw^{1-\gamma}\left(r_f + \alpha\mu - \frac{c}{w}\right)dt - \gamma(1-\gamma)\frac{1}{2}Aw^{1-\gamma}\alpha^2\sigma^2 dt$$

$$\Rightarrow r_f + \alpha\mu - \frac{c}{w} - \frac{\gamma}{2}\alpha^2\sigma^2 = 0$$

This expression is equal to drift of log wealth (and hence log consumption) only when $\gamma = 1$. But in general

$$r_f + \alpha\mu - \frac{c}{w} - \frac{1}{2}\alpha^2\sigma^2 - \frac{1}{2}(\gamma - 1)\alpha^2\sigma^2 = 0$$

$$r_f + \alpha\mu - \frac{c}{w} - \frac{1}{2}\alpha^2\sigma^2 = \frac{1}{2}(\gamma - 1)\alpha^2\sigma^2$$

so that optimal log wealth drift is positive for $\gamma > 1$ exactly as we found above.

The final step is to maximize consumption defined that is achieved by picking $\alpha = \frac{\mu}{\gamma\sigma^2}$ – a standard portfolio choice rule.

Relating to Sustainable Consumption Constraint Arithmetic average model implies that $\frac{c}{w} = r_f + \alpha\mu$ which combined with the $\psi \to 0$ restriction gives

$$r_f + \alpha\mu - (r_f + \alpha\mu) = \frac{\gamma}{2}\alpha^2\sigma^2$$

For $\gamma \geq 1$ this implies $\alpha = 0$.

Geometric average model implies that $\frac{c}{w} = r_f + \alpha\mu - \frac{1}{2}\alpha^2\sigma^2$. Combine this with the same constraint to get

$$r_f + \alpha\mu - \left(r_f + \alpha\mu - \frac{1}{2}\alpha^2\sigma^2\right) = \frac{\gamma}{2}\alpha^2\sigma^2 \Rightarrow \alpha^2 = \gamma\alpha^2$$

One solution is $\alpha = 0$. For $\gamma = 1$ the constraint sustainable consumption constraint doesn't bind since it coincides with Leontief "constraint". Therefore, the agent simply maximizes consumption by choosing the growth optimal portfolio $\alpha = \frac{\mu}{\sigma^2}$.

4 A Dynamic Model

HJB Equation for Multiple States Before going into the numerical solution we provide some details on the HJB equation with multiple dynamic constraints. The general problem is

$$v(x_0) = \max_{c_t} E \int_0^\infty e^{-\rho t} u(x_t, c_t) dt$$
$$dx_t = f(x_t, c_t)dt + \sigma(x_t, c_t) dZ_t$$

where x_t is $N \times 1$, dZ_t is $M \times 1$ and c_t is $K \times 1$. First, we need to define a $N \times N$ matrix

$$\Sigma(x_t, c_t) = \sigma(x_t, c_t)\sigma(x_t, c_t)'$$

Using $\Sigma(x_t, c_t)$ we can write the HJB equation as

$$\rho v(x) = \max_c \left\{ u(x, c) + \sum_{i=1}^N \frac{\partial v}{\partial x_i} f(x, c) + \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N \frac{\partial^2 v}{\partial x_i \partial x_j} \Sigma_{ij}(x, c) \right\}$$

In the dynamic model presented in the main text we have the following problem

$$\max_{\alpha_t} E_0 \int_0^\infty e^{-\rho t} u(c_t) dt$$

$$\text{subject to } c_t = w_t \left(r_t + \alpha_t \mu - \frac{1}{2} \alpha_t^2 \sigma^2 \right)$$
$$\begin{pmatrix} dw_t \\ dr_t \end{pmatrix} = \begin{pmatrix} \frac{1}{2} w_t \alpha_t^2 \sigma^2 \\ \phi(r_t) \end{pmatrix} + \begin{pmatrix} w_t \alpha_t \sigma & 0 \\ \nu r_t \eta & \nu r_t \sqrt{1-\eta^2} \end{pmatrix} \begin{pmatrix} dZ_t^{(1)} \\ dZ_t^{(2)} \end{pmatrix}$$

Matrix Σ is

$$\Sigma \equiv \begin{pmatrix} w_t \alpha_t \sigma & 0 \\ \nu r_t \eta & \nu r_t \sqrt{1-\eta^2} \end{pmatrix} \begin{pmatrix} w_t \alpha_t \sigma & \nu r_t \eta \\ 0 & \nu r_t \sqrt{1-\eta^2} \end{pmatrix} = \begin{pmatrix} w_t^2 \alpha_t^2 \sigma^2 & w_t \alpha_t \nu r_t \eta \\ w_t \alpha_t \nu r_t \eta & \nu^2 r_t^2 \end{pmatrix}$$

resulting in the HJB equation stated in the main text.

Problem Our goal is to numerically solve the following system of equations

$$\left(r + \alpha^* \mu - \frac{1}{2} (\alpha^*)^2 \sigma^2 \right)^{-\gamma} (\mu - \alpha^* \sigma^2) + A(r)(1-\gamma)\alpha^* \sigma^2 + A'(r) \nu r \eta = 0$$

$$\rho A(r)\frac{1}{1-\gamma} = \frac{\left(r+\alpha^*\mu-\frac{1}{2}(\alpha^*)^2\sigma^2\right)^{1-\gamma}}{1-\gamma} + A(r)\frac{1}{2}(1-\gamma)(\alpha^*)^2\sigma^2 + A'(r)\frac{1}{1-\gamma}\frac{1}{2}\nu^2 r + \frac{1}{2}A''(r)\frac{1}{1-\gamma}\nu^2 r^2 + A'(r)\alpha^*\nu r\eta$$

where HJB features the optimal risky share α^* derived from the FOC.

Discretization We first discretize the state space $r = r_1, \ldots, r_I$ with equidistant intervals so that $r_i - r_{i-1} = \Delta r\ \forall i$. To simplify notation denote $A(r_i) = A_i$. Denote the solution to the FOC for aprticular level of the interest rate r_i as α_i. We approximate the derivatives as follows

$$(A')_i \approx= \frac{A_{i+1} - A_{i-1}}{2\Delta r}$$

$$(A'')_i \approx \frac{A_{i+1} - 2A_i + A_{i-1}}{(\Delta r)^2}$$

Using these approximations the FOC becomes

$$0 = \left(r_i + \alpha_i\mu - \frac{1}{2}\alpha_i^2\sigma^2\right)^{-\gamma}(\mu - \alpha_i\sigma^2) + A_i(1-\gamma)\alpha_i\sigma^2 + \frac{A_{i+1} - A_{i-1}}{2\Delta r}\nu r_i\eta \tag{11}$$

and the discretized HJB equation then becomes (multiplied by $1-\gamma$)

$$\rho A_i = \left(r_i + \alpha_i\mu - \frac{1}{2}\alpha_i^2\sigma^2\right)^{1-\gamma} + A_i\frac{1}{2}(1-\gamma)^2\alpha_i^2\sigma^2 + \frac{A_{i+1} - A_{i-1}}{2\Delta r}\frac{1}{2}\nu^2 r + \frac{1}{2}\frac{A_{i+1} - 2A_i + A_{i-1}}{(\Delta r)^2}\nu^2 r_i^2 + \frac{A_{i+1} - A_{i-1}}{2\Delta r}(1-\gamma)\alpha_i r_i\nu\eta$$

collecting the terms we get

$$\rho A_i = \left(r_i + \alpha_i\mu - \frac{1}{2}\alpha_i^2\sigma^2\right)^{1-\gamma} + A_{i-1}\left[\frac{\nu^2 r_i^2}{2(\Delta r)^2} - \frac{(1-\gamma)\alpha_i r_i\nu\eta}{2\Delta r} - \frac{1}{2\Delta r}\frac{\nu^2 r_i}{2}\right] + A_i\left[\frac{1}{2}(1-\gamma)^2\alpha_i^2\sigma^2 - \frac{\nu^2 r_i^2}{(\Delta r)^2}\right] + A_{i+1}\left[\frac{\nu^2 r_i^2}{2(\Delta r)^2} + \frac{(1-\gamma)\alpha_i r_i\nu\eta}{2\Delta r} + \frac{1}{2\Delta r}\frac{\nu^2 r_i}{2}\right]$$

We impose the “reflecting barrier” constraints $A_0 = A_1, A_{I+1} = A_I$. Under these constraints the equation for $i = 1$ and $i = I$ becomes

$$\rho A_1 = \left(r_1 + \alpha_1\mu - \frac{1}{2}\alpha_1^2\sigma^2\right)^{1-\gamma} + A_1\left[\frac{1}{2}(1-\gamma)^2\alpha_1^2\sigma^2 - \frac{\nu^2 r_1^2}{2(\Delta r)^2} - \frac{(1-\gamma)\alpha_1 r_1\nu\eta}{2\Delta r} - \frac{1}{2\Delta r}\frac{\nu^2 r_1}{2}\right] + A_2\left[\frac{\nu^2 r_1^2}{2(\Delta r)^2} + \frac{(1-\gamma)\alpha_1 r_1\nu\eta}{2\Delta r} + \frac{1}{2\Delta r}\frac{\nu^2 r_1}{2}\right]$$

$$\rho A_I = \left(r_I + \alpha_I \mu - \frac{1}{2}\alpha_I^2 \sigma^2\right)^{1-\gamma} + A_{I-1}\left[\frac{\nu^2 r_I^2}{2(\Delta r)^2} - \frac{(1-\gamma)\alpha_I r_I \nu \eta}{2\Delta r} - \frac{1}{2\Delta r}\frac{\nu^2 r_I}{2}\right]$$
$$+ A_I \left[\frac{1}{2}(1-\gamma)^2 \alpha_I^2 \sigma^2 - \frac{\nu^2 r_I^2}{2(\Delta r)^2} + \frac{(1-\gamma)\alpha_I r_I \nu \eta}{2\Delta r} + \frac{1}{2\Delta r}\frac{\nu^2 r_I}{2}\right]$$

Now we write this in matrix notation to get

$$\begin{aligned} x_i &= \frac{\nu^2 r_i^2}{2(\Delta r)^2} - \frac{(1-\gamma)\alpha_i r_i \nu \eta}{2\Delta r} - \frac{1}{2\Delta r}\frac{\nu^2 r_i}{2} \\ y_i &= \frac{1}{2}(1-\gamma)^2 \alpha_i^2 \sigma^2 - \frac{\nu^2 r_i^2}{(\Delta r)^2} \\ z_i &= \frac{\nu^2 r_i^2}{2(\Delta r)^2} + \frac{(1-\gamma)\alpha_i r_i \nu \eta}{2\Delta r} + \frac{1}{2\Delta r}\frac{\nu^2 r_i}{2} \end{aligned} \Rightarrow B^n = \begin{pmatrix} y_1 + x_1 & z_1 & 0 & 0 & \cdots \\ x_2 & y_2 & z_2 & 0 & \cdots \\ 0 & x_3 & y_3 & z_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \cdots \\ 0 & \cdots & 0 & x_I & y_I + z_I \end{pmatrix}$$

where n denotes the iteration step and B^n emphasizes that it is calculated using α_i^n that is itself calculated using $\mathbf{A}^n$. Using this notation we can write the iteration as

$$\frac{\mathbf{A}^{n+1} - \mathbf{A}^n}{\Delta} + \rho \mathbf{A} = \mathbf{u}^n + B^n \mathbf{A}$$

The explicit method is

$$\frac{\mathbf{A}^{n+1} - \mathbf{A}^n}{\Delta} + \rho \mathbf{A}^n = \mathbf{u}^n + B^n \mathbf{A}^n \Rightarrow \mathbf{A}^{n+1} = \mathbf{A}^n + \Delta\left(\mathbf{u}^n + B^n \mathbf{A}^n - \rho \mathbf{A}^n\right)$$

However, the **implicit** method has better convergence properties:

$$\frac{\mathbf{A}^{n+1} - \mathbf{A}^n}{\Delta} + \rho \mathbf{A}^{n+1} = \mathbf{u}^n + B^n \mathbf{A}^{n+1} \Rightarrow \mathbf{A}^{n+1} = \left(\left(\frac{1}{\Delta} + \rho\right)\mathrm{eye}(I) - B^n\right)^{-1}\left(\mathbf{u}^n + \frac{1}{\Delta}\mathbf{A}^n\right)$$

Even though this method requires matrix inversion at every step of the iteration, the matrix is sparse and can be inverted efficiently using appropriate routines (e.g. available in Matlab or Julia).

Numerical Algorithm To sum up the algorithm is the following

1. Given $\mathbf{A}^n$ numerically solve for a vector of α_i^n using the first order condition in equation (11)
2. Given a vector α_i^n form vector $\mathbf{u}$ and matrix B^n
3. Update $\mathbf{A}$ using implicit scheme
$$\mathbf{A}^{n+1} = \left(\left(\frac{1}{\Delta} + \rho\right)\mathrm{eye}(I) - B^n\right)^{-1}\left(\mathbf{u}^n + \frac{1}{\Delta}\mathbf{A}^n\right)$$
4. Iterate until the difference between $\mathbf{A}^n$ and $\mathbf{A}^{n+1}$ becomes small, say less than 10^{-6}.