Portfolio Choice with Sustainable Spending: A Model of Reaching for Yield

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Abstract

We show that reaching for yield—a tendency to take more risk when the real interest rate declines while the risk premium remains constant—results from imposing a sustainable spending constraint on an otherwise standard infinitely lived investor with power utility. This is true for two alternative versions of the constraint which make wealth and consumption follow martingales in levels or in logs, respectively. Reaching for yield intensifies when the interest rate is initially low, helping to explain the salience of the topic in the current low-rate environment. The sustainable spending constraint also affects the response of risktaking to a change in the risk premium, which can even be negative when the riskless interest rate is sufficiently low. In a variant of the model where the sustainable spending constraint is formulated in nominal terms, low inflation also encourages risktaking.

1 Introduction

How does the level of the safe real interest rate affect investors' willingness to take risk? The conventional answer in finance theory, derived by Merton (1969, 1971), is that it does not. A long-lived investor with constant relative risk averse power utility, facing a constant risk premium, allocates a constant share of wealth to a risky asset regardless of the level of the real interest rate.[2]

Contrary to this conventional theory, the prolonged period of low interest rates in the last decade has led some observers to believe that investors "reach for yield", taking more risk when the real interest rate declines. Reaching for yield has been a particular concern among central bankers who fear that risk-taking may be an unintended consequence of loose monetary policy (Rajan 2006, 2013, Borio and Zhu 2012, Stein 2013).

Most of the literature on reaching for yield studies the responses to interest rates of financial intermediaries such as banks, insurance companies, money market funds, and pension funds.[3] To the extent that models of reaching for yield are offered, they rely on institutional frictions specific to these intermediaries such as bank liquidity management (Drechsler, Savov, and Schnabl 2018, Acharya and Naqvi 2019), insurance company duration management (Ozdagli and Wang 2019), a zero lower bound on the nominal return that banks and money funds can offer depositors (Chodorow-Reich 2014, Di Maggio and Kacperczyk 2017), or the discounting of pension fund liabilities when calculating funding status (Andonov, Bauer, and Cremers 2017, Lu et al 2019). An exception to this focus on intermediaries is Lian, Ma, and Wang (2019) who document reaching for yield among

[2]Merton's analysis, like this paper, is set in continuous time. Campbell (2018) provides a textbook exposition of the result in discrete time and shows how it extends from power utility to the related model where an investor has Epstein-Zin preferences.

[3]See for example Maddaloni and Peydró (2011), Chodorow-Reich (2014), Jiménez et al (2014), Hanson and Stein (2015), Andonov, Bauer, and Cremers (2017), Di Maggio and Kacperczyk (2017), Lu et al (2019), and Ozdagli and Wang (2019). There are also papers that use the term "reaching for yield" in a different sense, to refer to an unconditional propensity for financial intermediaries to take risk (Becker and Ivashina 2015, Choi and Kronlund 2017). In this paper we use the term exclusively to refer to the response of risk-taking to the interest rate.

households and suggest a behavioral explanation.[4]

In this paper we present a simple theoretical model that generates reaching for yield through a different mechanism. Our model applies most naturally to endowments and trusts, a different type of institutional investor than has been considered in the existing literature.

We start from the classic Merton model of an infinitely lived investor with power utility who chooses consumption and the portfolio allocation to a risky asset and a safe asset, both of which have iid returns. To this classic model we add one constraint, that the investor must consume in each period the expected return on the portfolio. This level of spending ensures that both consumption and wealth follow martingales, so the investor cannot plan either to run down wealth or to accumulate it. For this reason we call the constraint a "sustainable spending" constraint.

One natural interpretation of the sustainable spending constraint is that it reflects a promise that institutions with endowments, such as universities, make to donors. A gift to the endowment implies a commitment by the university to undertake an activity not just for a few years, but permanently. Donors who wish to have a greater short-term impact can make current-use gifts, but endowment gifts reflect a donor's desire for permanent impact.[5] Spending more than the expected return would run down the endowment and support only temporary spending, whereas spending less than the expected return would imply that the gift has less immediate impact than is needed for sustainability. In addition, spending less than the expected return would cause the endowment to grow on average, which can lead to unwelcome attention and even unfavorable tax

[4] There is a related literature on "reaching for income" among investors who value current income rather than total return (Hanson and Stein 2015). Daniel, Garlappi, and Xiao (2019) and Hartzmark and Solomon (2019) document such behavior among households when interest rates are low.

[5] As Harvard University puts it, "The University's spending practice has to balance two competing goals: the need to fund the operating budget with a stable and predictable distribution, and the obligation to maintain the long-term value of endowment assets after accounting for inflation." (https://www.harvard.edu/about-harvard/harvard-glance/endowment, accessed January 28, 2020.) A literature on endowment spending has discussed smoothing rules that stabilize the operating budget but has paid less attention to the long-run spending rate. See for example Dybvig (1995), Brown et al (2014), and Gilbert and Hrdlicka (2015). Here we focus on the determination of the long-run spending rate and its interaction with asset allocation.

consequences in the long run.[6]

The sustainable spending constraint is also a way to model the legal restrictions on endowments. Under the Uniform Prudent Management of Institutional Funds Act (UPMIFA), which governs endowment expenditures for nonprofit and charitable organizations in all US states except Pennsylvania, endowment spending must be consistent with the long-run preservation of capital but need not keep the endowment in any particular relation to the original historic value of gifts as was required under earlier state laws. Similar restrictions have been imposed on some sovereign wealth funds such as the Norwegian oil fund.[7]

The sustainable spending constraint may also describe the legal restrictions on trustees in cases where the principal of a trust is reserved for one beneficiary but income is paid to another. Under the 1997 revision of the Uniform Principal and Income Act, trustees are no longer bound to distribute only dividends or coupon payments to income beneficiaries, but have the "power to adjust" income distributions to reflect the expected return on investments (Sitkoff and Dukeminier 2017, pp. 669–670). If a trustee invests in the interests of a long-lived income beneficiary (that is, if the date of principal transfer is in the distant future), then the trustee's decisions may be well approximated by the model developed here.

Finally, the sustainable spending constraint might be interpreted more broadly as a parsimonious way to describe the behavior of older households who live off financial wealth but fear the consequences of running down that wealth, for example because they wish to preserve funds to

[6] The tax reform passed by Congress in 2017 includes an excise tax on private universities with over 500 tuition-paying students and assets of over $500,000 per student.

[7] Norges Bank Investment Management, the manager of Norway's Government Pension Fund Global (the official name of the oil fund) explains the restrictions in these words: "So that the fund benefits as many people as possible in the future too, politicians have agreed on a fiscal rule which ensures that we do not spend more than the expected return on the fund. On average, the government is to spend only the equivalent of the real return on the fund, which is estimated to be around 3% per year. In this way, oil revenue is phased only gradually into the economy. At the same time, only the return on the fund is spent, and not the fund's capital." (https://www.nbim.no/en/the-fund/about-the-fund/, accessed January 26, 2020.) The 3% figure is a current number; 4% was assumed until the spring of 2017.

cover unexpected medical expenses or to leave a bequest. Such households may be willing to take investment risk but may be unwilling to adopt a consumption plan that implies expected declines in future wealth.

The sustainable spending constraint breaks the separation between consumption/saving and asset allocation decisions that underlies the classic Merton result. When the interest rate declines, an investor with a constant rate of time preference wants to increase the expected growth rate of marginal utility. In the classic model, the investor achieves this by consuming more today and less in the future, but the sustainable spending constraint makes this impossible. Instead, the sustainable-spending constrained investor reaches for yield, taking more risk as a way to increase consumption today while paying the cost—volatile future consumption—only in the future.

We find that reaching for yield intensifies when the interest rate is initially low. This may help to explain why the phenomenon is more widely discussed today than it was in the higher-rate environment of the late 20th Century. The sustainable spending constraint also affects the response of risktaking to a change in the risk premium. This response declines as the interest rate declines and even becomes negative at sufficiently low levels of interest rates.

A subtle issue that we discuss in this paper is whether the expected return that governs sustainable spending is the expected simple (arithmetic average) return or the expected log (geometric average) return. In the former case, the levels of wealth and consumption are martingales while in the latter case the logs of wealth and consumption are martingales. The former case may seem more natural at first, but it implies that wealth and consumption approach zero with ever-increasing probability over time. As Dybvig and Qin (2019) emphasize, this is inconsistent with the spirit of the sustainable-spending constraint. Therefore, while we explore both cases in this paper, we emphasize the results for the geometric average case.

An important precursor to our work is a series of papers by Rampini and Viswanathan (2010, 2013, 2019). The first two of these papers point out that when firms hedge risk, they must post collateral. If firms also need collateral to finance profitable investments, corporate risk management becomes an intertemporal decision. The third paper points out that when households buy insurance, they must pay a premium up front. If households also face borrowing constraints, household risk management becomes an intertemporal decision. In these models, corporations or households take more risk when they face a tighter intertemporal constraint, just as investors do in our model. The difference is that our investors are constrained by a sustainable spending rule rather than by a collateral requirement, and the sustainable spending rule binds more tightly when the safe real interest rate is low.

The organization of the paper is as follows. In section 2 we set up a standard consumption and portfolio choice problem for an infinitely lived investor with power utility facing iid returns. We state the standard solution without a sustainable spending constraint, and contrast it with the solutions we obtain when we impose either an arithmetic or a geometric average sustainable-spending constraint. We derive a closed-form solution for the arithmetic case, and properties of the solution for the geometric case. To build intuition, we describe the constrained solutions using a mean-standard deviation diagram for the level and volatility of consumption. We also discuss the welfare costs of sustainable spending constraints.

Section 3 explores several extensions of the basic model. We consider what happens if the constraint is one-sided, preventing wealth decumulation but not accumulation; if the investor receives a stream of gifts; if the sustainable-spending constraint is on nominal spending in an environment with inflation; or if the investor has Epstein-Zin preferences.

Sections 2–3 present comparative static results, comparing economies with different constant levels of interest rates. In section 4, by contrast, we consider a dynamic model where the riskless interest rate follows a persistent stochastic process while the risk premium remains constant. In this model the portfolio rule depends on the correlation between the risky return and the innovation to the riskless interest rate, as in Merton (1973), Campbell and Viceira (2001), and other models of intertemporal hedging demand for risky assets. Section 5 concludes, and an online appendix (Campbell and Sigalov 2020) contains additional details of the analysis.

2 Comparative Statics with Power Utility

2.1 The Standard Unconstrained Model

We begin by writing down the standard model of consumption and portfolio choice for an infinitely lived investor with power utility facing iid returns, as presented for example in Ingersoll (1987, Chapter 13). The investor chooses consumption and the portfolio share in a risky asset to maximize the objective function

$$\mathrm{E}_0 \int_0^\infty e^{-\rho t} \frac{c_t^{1-\gamma}}{1-\gamma} dt \ , \tag{1}$$

subject to the intertemporal budget constraint

$$dw_t = w_t dr_{p,t} - c_t dt. \tag{2}$$

The portfolio return $dr_{p,t}$ is given by

$$dr_{p,t} = \alpha_t dr_t + (1 - \alpha_t) r_f dt, \tag{3}$$

where α_t is the portfolio share in a risky asset whose return dr_t is

$$dr_t = (r_f + \mu)dt + \sigma dZ_t. \tag{4}$$

Here r_f is the riskfree interest rate, μ is the risk premium, and σ is the volatility of the risky asset return. The Sharpe ratio of the risky asset is the ratio μ/σ. All these quantities are constant over time.

Combining these equations we can write the budget constraint as

$$\frac{dw_t}{w_t} = (r_f + \alpha_t \mu)dt + \alpha_t \sigma dZ_t - \frac{c_t}{w_t} dt. \tag{5}$$

We write the value function for this problem as $v(w_t)$. The Hamilton-Jacobi-Bellman (HJB) equation for the problem takes the form

$$\rho v(w_t) = \max_{\alpha_t, c_t} \left\{ \frac{c_t^{1-\gamma}}{(1-\gamma)} + v'(w_t) w_t \left(r_f + \alpha_t \mu - \frac{c_t}{w_t} \right) + \frac{1}{2} v''(w_t)^2 w_t^2 \alpha_t^2 \sigma^2 \right\}. \tag{6}$$

We guess that the value function takes the form

$$v(w_t) = A \frac{w_t^{1-\gamma}}{1-\gamma}, \tag{7}$$

for some unknown parameter A. Substituting this guess into the HJB equation, we have

$$\rho A \frac{w_t^{1-\gamma}}{1-\gamma} = \max_{\alpha_t, c_t} \left\{ \frac{c_t^{1-\gamma}}{(1-\gamma)} + A w_t^{1-\gamma} \left(r_f + \alpha_t \mu - \frac{c_t}{w_t} \right) - \frac{1}{2} \gamma A w_t^{1-\gamma} \alpha_t^2 \sigma^2 \right\}. \tag{8}$$

The first-order condition for α_t gives the usual formula for a constant risky share,

$$\alpha_t = \alpha = \frac{\mu}{\gamma \sigma^2}. \tag{9}$$

Portfolio volatility, $\alpha\sigma$, depends on the risk premium and volatility of the risky asset only through the Sharpe ratio (μ/σ). This is because leverage that alters the volatility of the risky asset without changing its Sharpe ratio does not affect the investor's opportunity set and hence does not alter the properties of the optimal portfolio or of consumption.

The first-order condition for consumption gives

$$c_t = A^{-\frac{1}{\gamma}} w_t. \tag{10}$$

We can substitute equations (9) and (10) into the HJB equation (8) to solve for A. The resulting solution implies that the consumption-wealth ratio is constant:

$$\frac{c_t}{w_t} = A^{-\frac{1}{\gamma}} = \frac{\rho}{\gamma} + \frac{\gamma - 1}{\gamma} \left(r_f + \frac{1}{2\gamma} \left(\frac{\mu}{\sigma} \right)^2 \right). \tag{11}$$

In the special case of log utility where $\gamma = 1$, the consumption-wealth ratio equals the rate of time preference ρ. In general however it depends also on investment opportunities. We are particularly interested in the case where $\gamma > 1$, in which case a higher riskfree rate and a higher Sharpe ratio on the risky asset increase the consumption-wealth ratio because the income effect of improved investment opportunities dominates the substitution effect.

Finally, we can substitute the solutions (9) and (11) into equation (5) to derive the stochastic processes for consumption and wealth. We find that

$$\frac{dw_t}{w_t} = \frac{dc_t}{c_t} = \left(\frac{r_f - \rho}{\gamma} + \frac{1 + \gamma}{2\gamma^2} \left(\frac{\mu}{\sigma} \right)^2 \right) dt + \frac{1}{\gamma} \left(\frac{\mu}{\sigma} \right) dZ_t. \tag{12}$$

This equation shows that in the usual consumption and portfolio choice problem, consumption and wealth grow at the same constant average rate that is determined by the riskfree rate, the rate of time preference, risk aversion, and the Sharpe ratio on the risky asset. We now consider what happens in a constrained problem where the investor is not free to choose the average growth rate of wealth.

2.2 An Arithmetic Sustainable Spending Constraint

Definition We first consider an arithmetic sustainable spending constraint. This is defined as the requirement that the flow of consumption equals the investor's wealth times the simple expected return on the agent's portfolio:

$$c_t dt = w_t \mathrm{E}_t dr_{p,t} = w_t (r_f + \alpha\mu) dt, \tag{13}$$

where the second equality assumes (and we later verify) that in the constrained problem the risky portfolio share α is constant.

If we substitute this consumption rule into the intertemporal budget constraint, equation (2), it implies that

$$dw_t = w_t \alpha\sigma dZ_t. \tag{14}$$

That is, wealth follows a martingale with volatility proportional to its level. The arithmetic constraint prevents the investor from planning to run down or to accumulate wealth over time. If, as we now show, consumption is a constant fraction of wealth, it too follows a martingale with volatility proportional to its level: the investor cannot plan for expected future consumption greater or less than current consumption.

Maximization problem Under the arithmetic constraint the maximization problem of the agent is

$$\max_{\alpha_t} \int_0^\infty e^{-\rho t} \frac{c_t^{1-\gamma}}{1-\gamma} dt \ = \int_0^\infty e^{-\rho t} \frac{(w_t(r_f + \alpha\mu))^{1-\gamma}}{1-\gamma} dt \tag{15}$$

subject to equation (14).

We once again guess that the value function takes the form

$$v(w_t) = B\frac{w_t^{1-\gamma}}{1-\gamma}, \tag{16}$$

where B is a constant representing investment opportunities that depends on the riskfree rate and the Sharpe ratio of the risky asset. Equation (16) implies that the HJB equation takes the form

$$\begin{aligned} \rho B\frac{w_t^{1-\gamma}}{1-\gamma} &= \max_{\alpha}\left\{\frac{(w_t(r_f+\alpha\mu))^{1-\gamma}}{1-\gamma} + \frac{1}{2}(-\gamma B w_t^{-\gamma-1})w_t^2\alpha^2\sigma^2\right\} \\ &= w_t^{1-\gamma}\max_{\alpha}\left\{\frac{(r_f+\alpha\mu)^{1-\gamma}}{1-\gamma} - \frac{1}{2}\gamma B\alpha^2\sigma^2\right\}. \end{aligned} \tag{17}$$

The corresponding first-order condition is

$$(r_f+\alpha\mu)^{-\gamma}\mu - B\sigma^2\alpha = 0. \tag{18}$$

Closed-form solution Using equations (17) and (18) we can solve for the risky share α explicitly as

$$\alpha = \frac{-r_f+\sqrt{K}}{\mu(1+\gamma)}, \tag{19}$$

where

$$K = r_f^2 + 2\rho\left(\frac{1+\gamma}{\gamma}\right)\left(\frac{\mu}{\sigma}\right)^2. \tag{20}$$

This solution has several standard properties. Portfolio volatility, $\alpha\sigma$, depends on the risk premium and volatility of the risky asset only through the Sharpe ratio (μ/σ). Also, the risky share α declines with the volatility σ of the risky asset and with risk aversion γ when other parameters of the model are fixed (although it is not inversely proportional to σ^2 or γ as would be the case in the standard model).

The solution also has several nonstandard properties summarized in the following proposition.

Proposition 1 *(Comparative Statics for the Arithmetic Average Model). In the arithmetic average model, the risky share α has the following properties.*

1. *α is a decreasing and convex function of the riskfree rate r_f.*
2. *α is an increasing function of the rate of time preference ρ.*
3. *α is an increasing function of the risk premium μ when $r_f > 0$, and a decreasing function of μ when $r_f < 0$.*

Proof: See the online appendix, Campbell and Sigalov (2020).

The first property says that the investor reaches for yield, taking more risk when the riskfree rate is low. Furthermore, the risky share is convex in the riskfree rate, implying that reaching for yield is stronger when the interest rate is low. This result may help to explain why reaching for yield is so actively discussed in today's low-rate environment, and why this paper has been written in 2020 even though the technology to write it was already standard decades ago.

The second property says that a more impatient investor takes more risk. Intuitively, risktaking increases current consumption and the volatility that this implies for future consumption is heavily discounted by an impatient investor.

The third property says that the riskfree rate influences the response of risktaking to the risk premium, which has the same sign as the riskfree rate. This response is constant and positive in the standard model, but once a sustainable spending constraint is imposed an increase in the risk premium has offsetting income and substitution effects on risktaking. The income effect becomes more powerful at low levels of the riskfree rate, and the two effects offset one another when the riskfree interest rate is zero. With $r_f = 0$ the solution (19) becomes

$$\alpha = \frac{1}{\sigma}\sqrt{\frac{2\rho}{\gamma(1+\gamma)}} \tag{21}$$

for any positive risk premium μ, which depends positively on the rate of time preference and negatively on risk and risk aversion, but not on the level of μ.[8]

2.3 A Geometric Sustainable Spending Constraint

Motivation Although the arithmetic sustainable spending constraint is simple and intuitive, it has an important disadvantage emphasized by Dybvig and Qin (2019). Application of Ito's Lemma to equation (14) tells us that under an arithmetic sustainable spending constraint, log wealth follows the process

$$d\log(w_t) = -\frac{1}{2}\alpha^2\sigma^2 dt + \alpha\sigma dZ_t, \tag{22}$$

which has a negative drift. The solution to this equation is

$$w_t = w_0 \exp\left\{-\frac{1}{2}\alpha^2\sigma^2 t + \sigma_c Z_t\right\}. \tag{23}$$

When we take the limit of this expression as $t \to \infty$, using the fact that $Z_t/t \to 0$ almost surely, we conclude that $w_t \to 0$ almost surely; and since the consumption-wealth ratio is constant, $c_t \to 0$ almost surely. Such a property is undesirable since it implies that spending will eventually approach zero as the investor exhausts available wealth. Intuitively, the right-skewed distributions of the levels of wealth and consumption, with constant means and increasing variances as the horizon increases, imply that almost all probability mass approaches zero while the constant expectations

[8] With $r_f = 0$ and $\mu = 0$ the problem has no solution because no positive level of consumption is sustainable. We discuss existence of a solution further in section 2.4.

of future wealth and consumption are sustained by a vanishingly small number of scenarios in which wealth and consumption are extremely high.[9]

Definition As an alternative, we now consider a geometric sustainable spending constraint. The value of the investor's portfolio, without subtracting consumption, follows the process

$$\frac{dV_t}{V_t} \equiv (r_f + \alpha\mu)dt + \alpha\sigma dZ_t. \tag{24}$$

The geometric sustainable consumption rule is defined as the requirement that the flow of consumption equals the investor's wealth times the expected log return on the agent's portfolio:

$$\begin{aligned} c_t dt &= w_t \mathrm{E}[d\log V_t] \\ &= w_t \left[r_f + \alpha\mu - \frac{1}{2}\alpha^2\sigma^2 \right] dt, \end{aligned} \tag{25}$$

where the second line is obtained by applying Ito's lemma to equation (24).

If we substitute this consumption rule into the intertemporal budget constraint, equation (2), and simplify we obtain

$$dw_t = w_t \frac{1}{2}\alpha^2\sigma^2 dt + w_t \alpha\sigma dZ_t, \tag{26}$$

and using Ito's lemma once again we get

$$d\log(w_t) = \alpha\sigma dZ_t. \tag{27}$$

This shows that in the geometric average model, the log and not the level of wealth follows a martingale. We will show that the consumption-wealth ratio is constant, in other words that

[9] An alternative intuition is that with an arithmetic sustainable spending rule the levels of wealth and consumption follow martingales bounded below by zero. Like any bounded martingales, they must converge almost surely, but the only level to which they can converge is zero because this is the only level at which they have zero volatility. Martin (2012) presents a related analysis.

log consumption equals log wealth plus a constant, so log consumption also follows a martingale. Because log wealth and log consumption are conditionally normally distributed at all horizons, the median values of future consumption and wealth equal their current values. This behavior seems more in accord with the spirit of a sustainable spending constraint.

Maximization problem Under the geometric constraint the maximization problem of the agent is

$$\max_{\alpha_t} \int_0^\infty e^{-\rho t} \frac{c_t^{1-\gamma}}{1-\gamma} dt = \int_0^\infty e^{-\rho t} \frac{\left(w_t[r_f + \alpha\mu - \frac{1}{2}\alpha^2\sigma^2]\right)^{1-\gamma}}{1-\gamma} dt \tag{28}$$

subject to equation (26).

We guess that the value function takes the same form as before, but with a different scale factor C:

$$v(w_t) = C\frac{w_t^{1-\gamma}}{1-\gamma}. \tag{29}$$

This implies that the HJB equation for the geometric model is

$$\rho C \frac{w_t^{1-\gamma}}{1-\gamma} = w_t^{1-\gamma} \max_\alpha \left\{ \frac{(r_f + \alpha\mu - \frac{1}{2}\alpha^2\sigma^2)^{1-\gamma}}{1-\gamma} + \frac{1}{2}(1-\gamma)C\alpha^2\sigma^2 \right\}, \tag{30}$$

with first-order condition

$$\left(r_f + \alpha\mu - \frac{1}{2}\alpha^2\sigma^2\right)^{-\gamma} + (1-\gamma)C\alpha\sigma^2 = 0. \tag{31}$$

Unlike the arithmetic average model, the geometric average model does not admit a closed form solution. However, by combining equations (30) and (31) and using the implicit function theorem we can prove the following proposition.

Proposition 2 *(Comparative Statics for the Geometric Average Model). In the geometric average model, the risky share α has the following properties.*

1. *α is a decreasing and convex function of the riskfree rate r_f.*
2. *α is an increasing function of the rate of time preference ρ.*
3. *There exists $r_f^* < 0$ such that for $r_f > r_f^*$, α is an increasing function of the risk premium μ and for $r_f < r_f^*$, α is a decreasing function of μ.*
4. *As risk aversion γ approaches 1, α approaches the growth-optimal level μ/σ^2 for all values of r_f.*

Proof: See the online appendix, Campbell and Sigalov (2020).

The first two properties are the same as in Proposition 1. The third property is similar to the third property in Proposition 1, but in the geometric case the interest rate that causes a sign switch in the effect of the risk premium on risktaking is not necessarily zero as it is in the arithmetic case. The fourth property is new to the geometric case, and reflects the fact that with a geometric constraint the growth-optimal portfolio (which would be optimal in the standard model) also maximizes current consumption.

To illustrate these properties, Figure 1 plots the geometrically constrained optimal risky share as a function of the riskfree rate for three different values of the risk premium. The parameters assumed are a risk premium of 6% in the base case (varying up to 7% or down to 5%), an 18% standard deviation of the risky return, a 7.5% rate of time preference, and risk aversion of 3. These parameters imply that in the base case the Sharpe ratio of the risky asset is 1/3 and the unconstrained Merton solution for the risky share is $\alpha = 62\%$—close to the conventional 60% equity allocation that small endowments have often used in the past.

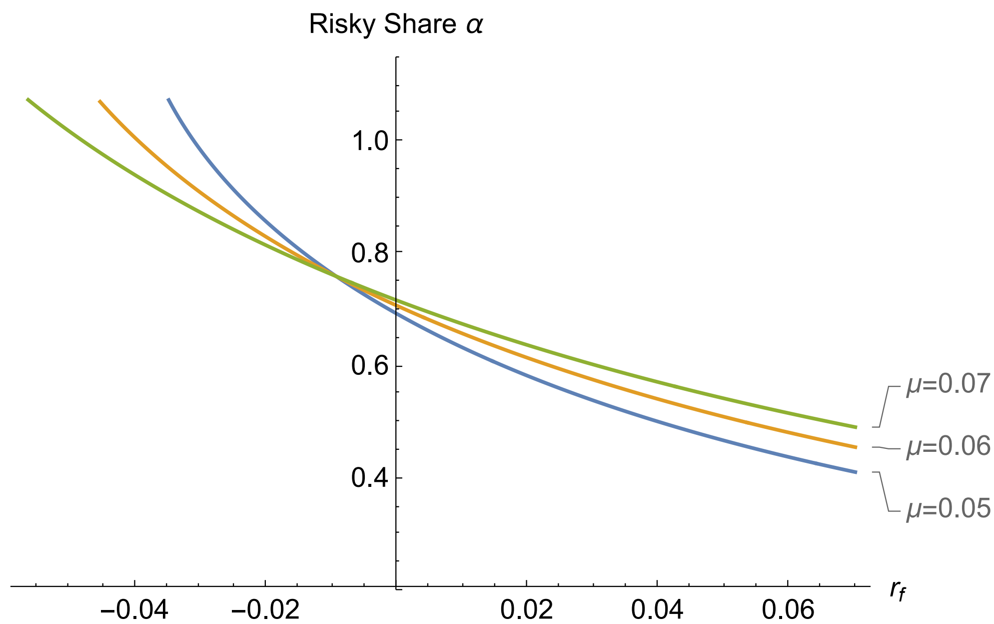

Figure 1: Sensitivity of Optimal Risky Share to Riskfree Rate and Risk Premium

The asset allocation curves plotted in Figure 1 are downward-sloping and convex, illustrating the first property in Proposition 2. These curves are flatter when the risk premium is higher, and they cross at a particular point with a negative riskfree interest rate. At this point the risky share is independent of the risk premium, while at higher values of the riskfree rate the risk premium has the usual positive effect on the risky share, and at lower (more negative) values of the riskfree rate the risk premium has a negative effect on the risky share. This illustrates the third property in Proposition 2.

An alternative graphical presentation may also be helpful. In the standard portfolio choice problem the optimal risky share is $\alpha = \mu/\gamma\sigma^2$ resulting in an expected (arithmetic average) portfolio return of

$$\mathrm{E}r_p = r_f + \alpha\mu = r_f + \frac{\mu^2}{\gamma\sigma^2}. \tag{32}$$

This implies that the derivative of the expected portfolio return with respect to the riskless interest rate is one. If we plot the expected portfolio return $\mathrm{E}r_p$ against r_f we get a 45-degree line with a positive intercept.

In the portfolio choice problem with an arithmetic or geometric sustainable spending constraint, a lower riskfree rate leads to a higher risky share. This implies that the derivative of the expected portfolio return with respect to the riskless interest rate is less than one. If we plot $\mathrm{E}r_p$ against r_f for the constrained problem, the resulting curve will be flatter than a 45-degree line.

We illustrate this property in Figure 2 where we plot the expected portfolio return against the riskfree rate r_f for several different portfolios.[10] The figure includes a 45-degree line going through the origin, representing a portfolio with $\alpha = 0$ that is fully invested in the riskfree asset. It also includes a 45-degree line with a positive intercept of μ^2/σ^2 corresponding to the growth-optimal portfolio that will be held by an investor with log utility. A conservative investor with $\gamma > 1$ and a geometric sustainable spending constraint picks a portfolio whose expected return lies between these two lines.

Figure 2 illustrates the optimally chosen expected return for geometrically constrained investors with $\gamma = 2$ (upper blue curve) and $\gamma = 3$ (lower orange curve). The slope of these curves is less than one, and for a more aggressive investor with $\gamma = 2$ the slope actually becomes negative when the riskfree rate becomes sufficiently low. In this region of the parameter space the tendency to reach for yield is so strong that a small decrease in the risk free rate can actually increase the arithmetic expected portfolio return (although it does not increase the geometric expected portfolio return and therefore does not increase the investor's current consumption).

[10] All parameters are the same as in the base case of Figure 1 except that we consider risk aversion levels of 1 and 2 as well as 3.

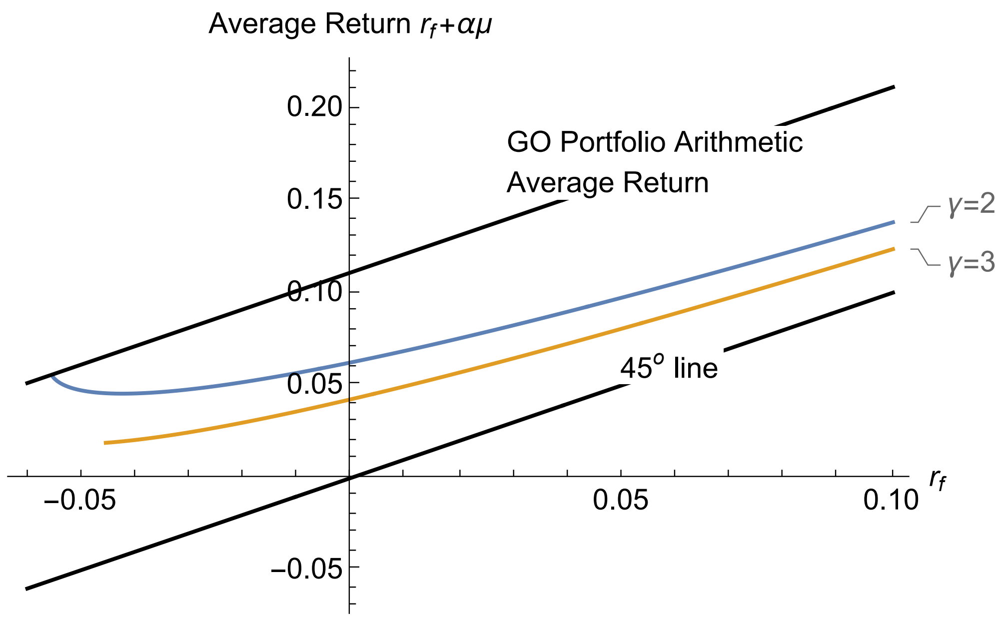

Figure 2: Reaching for Yield. Portfolio Expected Return, the Riskfree Rate, and Risk Aversion.

2.4 Mean-Standard Deviation Analysis for Consumption

To better understand infinite-horizon portfolio choice with sustainable spending constraints, we now characterize the problem as a tradeoff that the investor must make between the initial level and the volatility of consumption. This is analogous to the classic tradeoff between the mean and the volatility of portfolio return in single-period mean-variance analysis, but it differs from the standard unconstrained infinite-horizon problem where the investor can freely choose both the initial level and the volatility of consumption, which jointly determine the subsequent average growth rate of consumption.

Any constant risky share and consumption-wealth ratio in our unconstrained problem imply that the investor's log consumption follows a Brownian motion with an arbitrary constant drift μ_c and constant volatility σ_c:

$$d \log c_t = \mu_c dt + \sigma_c dZ_t. \tag{33}$$

A straightforward application of Ito's lemma tells us the stochastic process followed by the level of consumption:

$$dc_t = c_t\left(\mu_c + \frac{1}{2}\sigma_c^2\right)dt + c_t\sigma_c dZ_t. \tag{34}$$

Using this law of motion for consumption, we can derive the HJB equation describing the value function for an infinitely lived investor with time-separable utility and time discount rate ρ. Here we use consumption rather than wealth as the argument of the value function $v(c_t)$:

$$\rho v(c_t) = u(c_t) + v'(c_t)c_t\left(\mu_c + \frac{1}{2}\sigma_c^2\right) + \frac{1}{2}v''(c_t)^2\sigma_c^2. \tag{35}$$

We now assume that the investor has instantaneous power utility of consumption with relative risk aversion γ. Under this assumption the value function can be written as

$$v(c_t) = D\frac{c_t^{1-\gamma}}{1-\gamma} \tag{36}$$

for some parameter D. Substituting equation (36) into equation (35) and simplifying, we can solve for D as

$$D = \left(\rho + (\gamma-1)\mu_c - (\gamma-1)^2\frac{\sigma_c^2}{2}\right)^{-1}. \tag{37}$$

We will consider the empirically relevant case where the investor's coefficient of relative risk aversion $\gamma > 1$. In this case D in (37) decreases with μ_c and increases with σ_c; but it is multiplied by a negative number in (36) to deliver a negative value function that, as one would expect, increases with μ_c and decreases with σ_c.

An important implication of equation (37) is that there is an upper bound on the volatility of consumption that is consistent with finite utility of the infinitely lived investor. When σ_c exceeds this upper bound, D becomes negative and the value function is undefined.

Putting these results together, we have shown that the value function for a power-utility investor with a stream of consumption given by equation (33) is

$$v(c_0) = E_0 \int_0^\infty e^{-\rho t} u(c_t) dt = \left(\rho + (\gamma - 1)\mu_c - (\gamma - 1)^2 \frac{\sigma_c^2}{2} \right)^{-1} \frac{c_0^{1-\gamma}}{1-\gamma}. \tag{38}$$

If we fix the initial level of the value function at some constant v, then equation (38) can be rewritten as an indifference condition relating the initial level of consumption c_0 and the volatility of log consumption σ_c:

$$c_0 = \left[\left(\rho + (\gamma - 1)\mu_c - (\gamma - 1)^2 \frac{\sigma_c^2}{2} \right) (1-\gamma) v \right]^{\frac{1}{1-\gamma}}. \tag{39}$$

This equation tells us that c_0 is increasing in both σ_c and v: a larger initial level of consumption is required to compensate the investor for higher volatility given constant value, or to deliver higher value with constant volatility. Hence, if we plot indifference curves in (σ_c, c_0) space, each curve is upward sloping and the agent will pick the highest possible curve subject to a constraint that we next derive from the portfolio choice problem.

Sustainable spending rules determine the drift in log consumption, μ_c. They also impose constraints on the relationship between c_0 and σ_c, since an increase in consumption today can be financed only by taking more portfolio risk which implies greater volatility in consumption growth.

Arithmetic indifference condition and constraint An arithmetic sustainable spending rule implies that the drift in the level of consumption is zero: $\mu_c + \sigma_c^2/2 = 0$ in equation (34). In this case the indifference condition (39) becomes

$$c_0 = \left[\left(\rho - \gamma(\gamma - 1) \frac{\sigma_c^2}{2} \right) (1-\gamma) v \right]^{\frac{1}{1-\gamma}}. \tag{40}$$

Normalizing initial wealth $w_0 = 1$, the arithmetic rule implies that the relationship between the initial level of consumption and the volatility of consumption is

$$c_0 = r_f + \left(\frac{\mu}{\sigma}\right)\sigma_c, \tag{41}$$

where r_f is the riskless interest rate, μ is the expected excess return on a risky asset, and σ is the standard deviation of the risky asset return. Equation (41) follows from the familiar relationships in a Merton model that $\sigma_c = \alpha\sigma$ and the arithmetic expected portfolio return is $r_f + \alpha\mu$, where α is the portfolio share in a risky asset. It implies a linear tradeoff between the initial level and the volatility of consumption with intercept r_f and slope (μ/σ).

Geometric indifference condition and constraint Under the geometric sustainable spending rule, the indifference condition (39) becomes

$$c_0 = \left[\left(\rho - (\gamma-1)^2\frac{\sigma_c^2}{2}\right)(1-\gamma)v\right]^{\frac{1}{1-\gamma}}. \tag{42}$$

Comparing the geometric indifference condition (42) with the arithmetic indifference condition (40), the difference is in the coefficient multiplying $\sigma_c^2/2$. Since $\gamma(\gamma-1) > (\gamma-1)^2$ under our maintained assumption that $\gamma > 1$, for any given value v the indifference curve for the arithmetic rule lies above the indifference curve for the geometric rule, except when $\sigma_c = 0$ where the two curves converge.

Normalizing initial wealth $w_0 = 1$, the geometric rule implies that the relationship between the initial level of consumption and the volatility of consumption is

$$c_0 = r_f + \left(\frac{\mu}{\sigma}\right)\sigma_c - \frac{1}{2}\sigma_c^2, \tag{43}$$

where the last term is the Jensen's Inequality difference between the arithmetic and geometric mean of consumption growth. Equation (43) is a concave rather than a linear constraint. As volatility increases, it has a diminishing effect on the expected log portfolio return and therefore on the initial level of consumption.

The highest level of initial consumption is obtained when the investor holds the growth-optimal portfolio with maximum log return. In this case $\sigma_c = \mu/\sigma$ and

$$c_0 = r_f + \frac{1}{2}\left(\frac{\mu}{\sigma}\right)^2. \tag{44}$$

As in the standard Merton problem, we can show that as γ approaches one, the optimal portfolio approaches a growth optimal portfolio and consumption will be given by equation (44).

Graphical analysis To understand the properties of indifference curves with sustainable spending constraints, we can differentiate equations (40) or (42) with respect to the standard deviation of consumption, σ_c, to see that under either type of constraint, the slope of the indifference curve is positive and increasing in σ_c. The slope is zero—the indifference curve is flat—at zero standard deviation, reflecting the fact that investors with twice differentiable utility always take some amount of any compensated risk. The slope approaches infinity as the standard deviation approaches the upper bound at which the value function no longer converges. Since as already noted $\gamma(\gamma-1) > (\gamma-1)^2$ under our maintained assumption that $\gamma > 1$, the slope is everywhere greater and the upper bound is smaller for the arithmetic constraint than for the geometric constraint.

These properties are illustrated in Figure 3. There we plot indifference curves (40) and (42), and spending constraints (41) and (43), for both types of constraint and a common level of value v.[11] The two indifference curves have the same intercept with the vertical axis, but the arithmetic indifference curve is higher and steeper elsewhere and its asymptote lies to the left of that for the geometric indifference curve.

[11] The parameters assumed are a 2% riskfree interest rate together with the base case parameters from Figure 1.

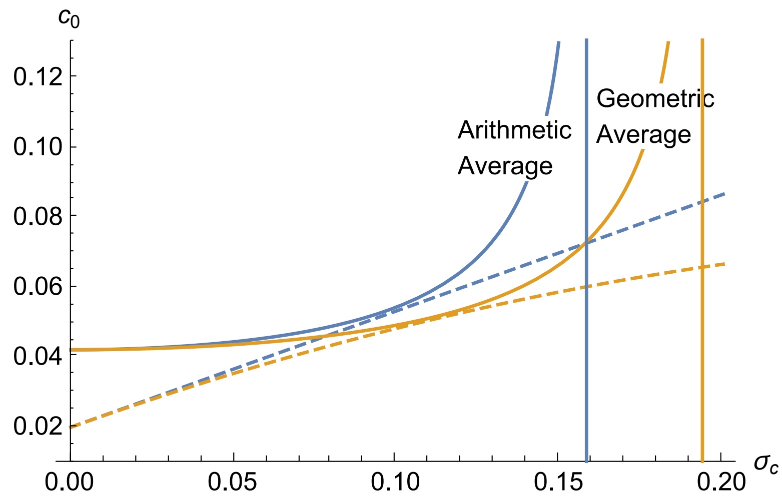

Figure 3: Comparing Arithmetic and Geometric Average Spending Constraints

Figure 3 also illustrates the fact that at the optimum for the geometric rule, that is for the value v where the geometric indifference curve is tangent to its concave portfolio constraint, the arithmetic indifference curve lies above its linear portfolio constraint. To obtain the optimal choice under the arithmetic constraint we need to shift the indifference curve down, corresponding to a lower value, in order to satisfy the portfolio constraint. In other words optimized consumption delivers lower lifetime utility under the arithmetic constraint than under the geometric constraint; this may be an additional reason to prefer the geometric formulation of the sustainable spending constraint as advocated by Dybvig and Qin (2019).

Existence of a solution Figure 3 allows us to understand the conditions for the existence of a solution: a point where an indifference curve is tangent to the spending constraint. First, since all indifference curves must be in the positive quadrant, a necessary condition for a tangency point to exist is that the spending constraint has a maximum above zero. This condition is trivially satisfied by the arithmetic constraint, which is linear with a positive slope, but not by the concave geometric constraint. If the riskfree rate is low enough, then even when the investor holds the growth-optimal portfolio, the expected portfolio return may be negative and no sustainable solution will exist for a geometric spending constraint. This problem arises if

$$r_f < -\frac{1}{2}\left(\frac{\mu}{\sigma}\right)^2. \tag{45}$$

Second, a solution may also fail to exist if the spending constraint becomes positive only at levels of consumption volatility σ_c that exceed the upper bound for the value function to be finite. This problem can arise either for the arithmetic constraint or for the geometric constraint. In the geometric case, from equation (42) the highest permissible level of consumption volatility is

$$\sigma_c^\star = \sqrt{\frac{2\rho}{(1-\gamma)^2}}, \tag{46}$$

and from equation (43) the portfolio constraint intersects the x-axis at the point

$$\sigma_c^{\star\star} = \mu/\sigma - \sqrt{\mu^2/\sigma^2 + 2r_f}. \tag{47}$$

Existence of a solution requires that $\sigma_c^\star > \sigma_c^{\star\star}$, which is guaranteed when $r_f > 0$ but may fail when $r_f < 0$.

Figure 2 illustrates these two different conditions for existence of a solution to the portfolio choice problem. For a relatively aggressive investor with risk aversion $\gamma = 1.5$, the first condition binds when the riskfree rate is very low. Below the binding value of the riskfree rate, even the growth-optimal portfolio does not provide positive consumption: in Figure 2 the upper blue curve starts from the growth-optimal optimal portfolio line.[12] For a more conservative investor with risk aversion $\gamma = 3$, on the other hand, the second condition binds: in Figure 2 the leftmost point of the lower orange curve is to the right of the leftmost point of the upper curve. A conservative investor

[12]Since the figure plots the arithmetic average portfolio return, the line is positive at this point but the geometric average portfolio return that enters the sustainable spending constraint is zero.

with a geometric average spending constraint cannot tolerate as low a riskfree interest rate as can a more aggressive investor with the same constraint.

Reaching for yield The graphical analysis can be used to understand why sustainable spending rules generate reaching for yield. A change in the riskfree interest rate, with no change in the risk premium, is a parallel shift up or down in the spending constraint. An increase in the riskfree rate is an upward shift that improves the opportunity set and increases the achievable value v, while a decrease is a downward shift that reduces v.

Equations (40) and (42) imply that for both the arithmetic and geometric cases, the slope of the indifference curve is increasing in v. Mathematically, this is because v enters multiplicatively in these equations, increasing both the value and the slope of the indifference curve for any value of consumption volatility.

The response of the slope of the indifference curve to the level of value v implies that as the spending constraint shifts up, the optimal standard deviation of consumption declines; while as the spending constraint shifts down, the optimal standard deviation of consumption increases. This is precisely reaching for yield. It is illustrated in Figure 4 for the geometric case.[13]

Reaching for yield reflects the fact that in the problem with a sustainable spending constraint both consumption today and the stability of the consumption path are normal goods, so the investor uses an improvement in investment opportunities to increase both of them, in other words to reduce volatility as well as to increase current consumption.

The effect of impatience We can also analyze the effect of impatience on the problem. Equations (40) and (42) imply that for both the arithmetic and geometric cases, the slope of the indifference curve is decreasing in the time discount rate ρ. A more impatient investor values current

[13] All parameters are the same as in Figure 3 except that we consider riskfree rates of 2.5%, 0%, and −2.5%.

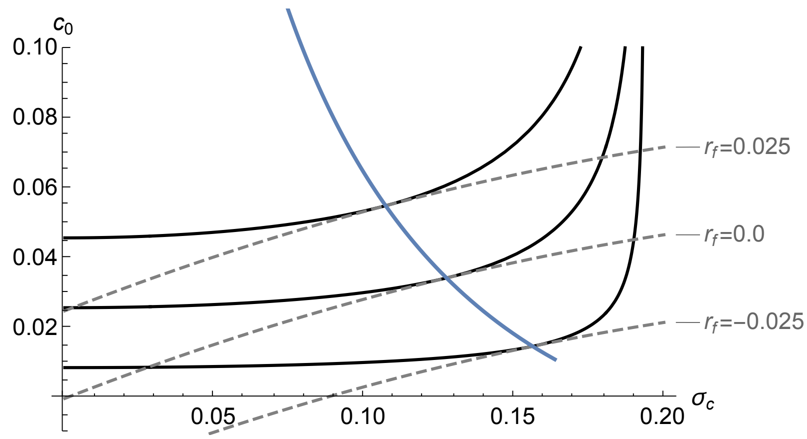

Figure 4: Reaching for Yield. The Effect of the Riskfree Rate on Initial Consumption and Consumption Volatility

consumption more relative to volatility that is realized in the future. Hence, the optimal solution will involve higher current consumption and higher consumption volatility as illustrated in Figure 5 for the geometric case.[14]

The effect of the risk premium In the standard Merton model a larger risk premium unambiguously raises the share of wealth allocated to risky assets. Propositions 1 and 2 show that this is no longer the case in the model with a sustainable spending constraint. As illustrated in Figure 1, when the riskfree interest rate is sufficiently low, a larger risk premium may actually reduce the risky share.

The reason is that there are two effects in play when the risk premium changes. An increase in the risk premium both shifts the portfolio constraint up, and makes it steeper. The former effect reduces risktaking and the latter increases it. As the riskfree rate declines, the former reaching for yield effect becomes more powerful and it dominates at sufficiently low levels of the riskfree rate.

[14] All parameters are the same as in Figure 3 except that we consider time preference rates of 5%, 7.5%, and 10%.

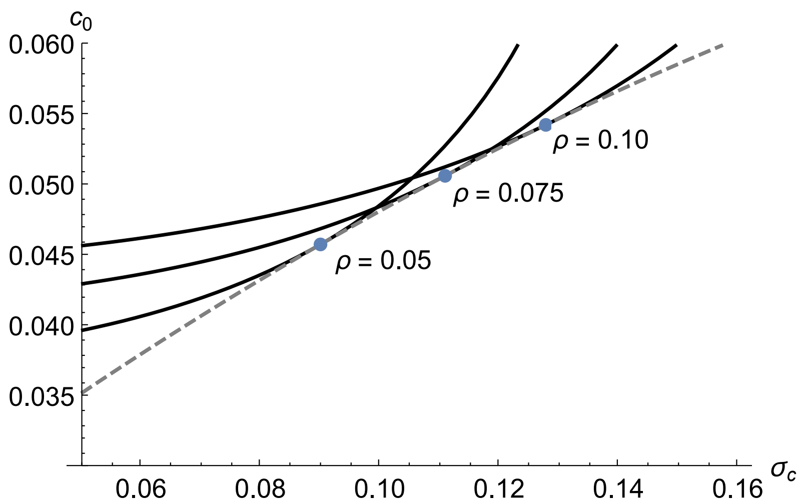

Figure 5: The Effect of Impatience on Initial Consumption and Consumption Volatility

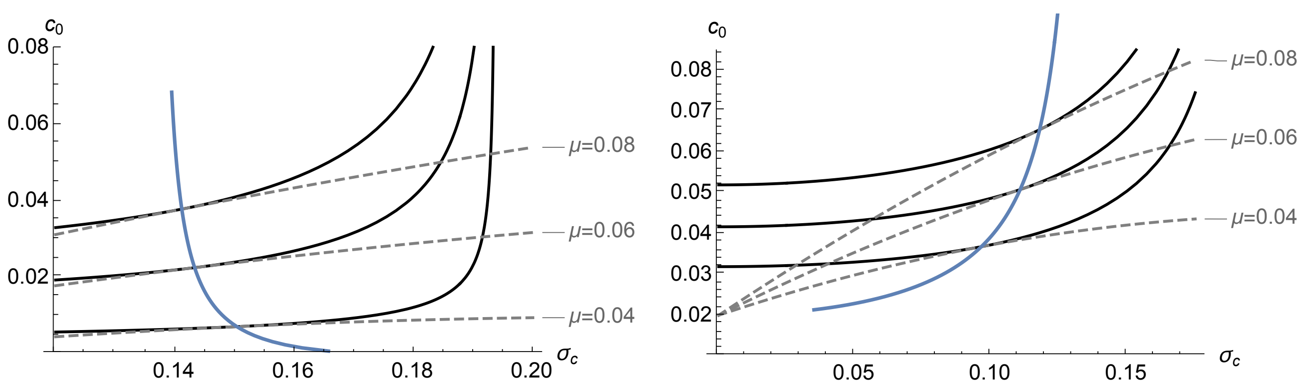

Figure 6: The Effect of Risk Premium on Initial Consumption and Consumption Volatility

Figure 6 has two panels illustrating the two possible situations.[15] The left panel shows a case where r_f is very low at -1.5%. Here an increase in value leads to a significant steepening of the indifference curves. This effect dominates the steepening of the portfolio constraint and results in lower consumption volatility for larger values of the risk premium. The right panel shows the case where r_f is 2%. Here an increase in value results in an almost parallel shift of the indifference curves that is dominated by the steepening of the portfolio constraint.

2.5 The Welfare Cost of Sustainable Spending

A sustainable spending constraint forces an investor to deviate from the optimal Merton rules for consumption and portfolio allocation. In this section we quantify the utility losses from this restriction, solving for the share of wealth ψ that the investor is willing to give up to remove the constraint. Using the closed-form expression for the investor's value v given the drift μ_c and volatility σ_c of log consumption and the consumption-wealth ratio in equation (38), we define ψ as

$$v(\mu_c, \sigma_c, c/w, (1-\psi)w_0) = v(\mu_c^*, \sigma_c^*, (c/w)^*, w_0), \tag{48}$$

where terms with asterisks denote the constrained solutions. Rearranging this expression and using (38) gives

$$\psi = 1 - \frac{\left(\rho + (\gamma-1)\mu_c^* - (\gamma-1)^2 \frac{(\sigma_c^*)^2}{2}\right)^{\frac{1}{\gamma-1}}}{\left(\rho + (\gamma-1)\mu_c - (\gamma-1)^2 \frac{\sigma_c^2}{2}\right)^{\frac{1}{\gamma-1}}} \frac{(c/w)^*}{(c/w)}. \tag{49}$$

There is a level of the riskfree interest rate at which the sustainable spending constraint does not bind, because the unconstrained agent would freely choose a random walk for the level or log of consumption and wealth. In the base case of Figure 1 with a geometric sustainable spending constraint, this level of the riskfree interest rate is very close to 2%, as one can see from the fact

[15] Risk aversion, the rate of time preference, and the volatility of the risky asset return are the same as in Figure 3.

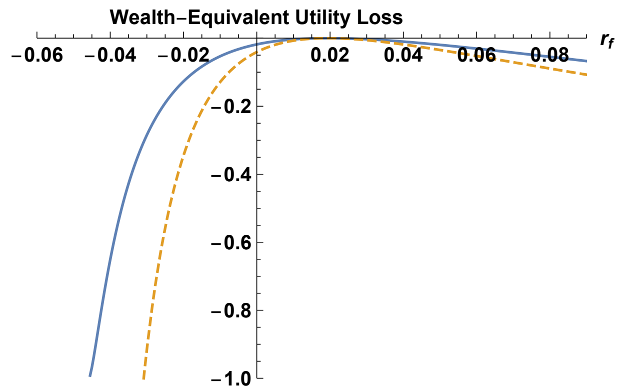

Figure 7: Wealth-Equivalent Welfare Cost of a Geometric Sustainable Spending Constraint

that the constrained risky share equals the unconstrained value of 62% when $r_f \approx 2\%$. At this point the welfare cost of the constraint $\psi = 0$.

We have discussed the fact that when the riskfree interest rate gets low enough, a solution with sustainable spending fails to exist. At this point the utility cost of the constraint is infinite and $\psi = 1$.

Figure 7 illustrates these properties for the base case of Figure 1. The solid blue line plots the wealth-equivalent welfare cost, $-\psi$, against the riskfree interest rate. There is a point close to $r_f = 2\%$ where the welfare cost of the constraint is zero, but as the riskfree rate falls the welfare loss mounts rapidly until the point where the solution ceases to exist. As the riskfree interest rate increases above 2%, the welfare loss also increases but not as rapidly due to the convexity of the risky share as a function of the riskfree interest rate.

This calculation may help us to understand why endowments have accepted restrictions on their flexibility to adjust spending. When interest rates are modestly positive, for the parameter values considered there is almost no welfare cost of a sustainable spending constraint since the constraint is consistent with the endowment's desired spending path. Thus, in the interest-rate environment of the late 20th Century, endowments may have found it natural to reassure donors about their spending intentions by agreeing to a sustainable spending constraint. They may not have anticipated the current environment of persistently low interest rates in which the constraint binds tightly and has serious welfare implications.

Welfare implications of reaching for yield We have shown that a sustainable spending constraint generates reaching for yield behavior. We now ask what is the welfare consequence of this behavior. If we impose a further restriction that the investor's asset allocation is fixed at the level prescribed by the Merton rule, we can recalculate the welfare cost ψ. This is shown in Figure 7 as an orange dashed line.

Naturally the additional restriction further lowers the investor's utility, and we see that the wealth-equivalent welfare cost of sustainable spending increases much more rapidly as the riskfree interest rate declines. For $r_f = -1\%$, the wealth-equivalent welfare cost of the sustainable spending constraint is 14% if reaching for yield is restricted but only 5% when reaching for yield is allowed; for an even lower $r_f = -2\%$ the welfare cost is 30% when reaching for yield is restricted and 10% when it is allowed.

This calculation may help us to understand why endowments have maintained control over their asset allocation even while accepting constraints on their spending behavior. If endowments historically believed that there was a small possibility of entering a persistently low-rate environment, they might have retained the ability to reach for yield as a way to limit the welfare consequences of persistently low interest rates given a sustainable spending constraint.

3 Extensions of the Static Model

3.1 A One-Sided Sustainable Spending Constraint

The sustainable spending constraint we have considered so far prohibits both accumulation and decumulation of wealth. An alternative type of constraint is one-sided, preventing decumulation but not accumulation of wealth. In this case the investor is unconstrained when the riskfree interest rate is high enough to induce wealth accumulation, but constrained at low levels of the riskfree rate that incentivize decumulation.

We show optimal risky shares for a one-sided constraint in Figure 8. All the parameters here are the same as in Figure 1, so the three curves are the same at low levels of the riskfree rate but flatten out to the Merton solutions at higher levels of the riskfree rate. The kinks occur at the points where the constraint stops binding: about $r_f = 0\%$ for a 7% risk premium, $r_f = 2\%$ for the base case of a 6% risk premium, and $r_f = 3.5\%$ for a 5% risk premium.

The possibility of one-sided constraints suggests an additional reason why reaching for yield has become a more salient topic in the current low-interest-rate environment. Some investors with one-sided constraints may now be constrained and reaching for yield, when they were not at the higher interest-rate levels of the late 20th Century.

3.2 Gifts

We can extend our analysis to consider an investor who spends not only out of financial wealth, but also out of gifts such as those alumni make to universities. For tractability, we assume that the value of gifts received is proportional to the existing level of wealth.

Gifts to endowments come in two main varieties. Current-use gifts can be spent in the period

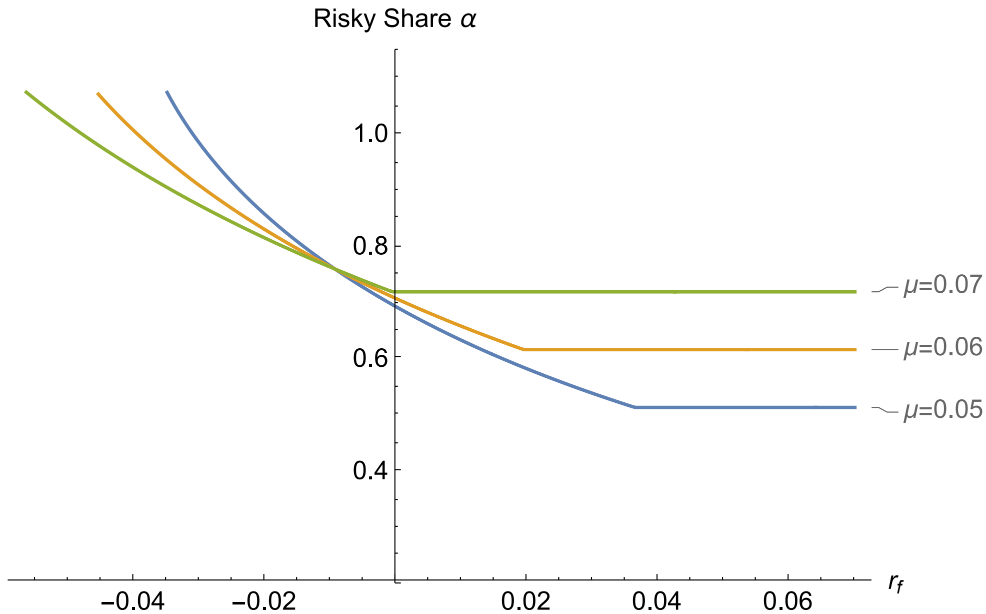

Figure 8: Sensitivity of Optimal Risky Share to Riskfree Rate and Risk Premium for One-Sided Constraint

in which they are received, while endowment gifts must be added to financial wealth and generate a stream of subsequent spending subject to any sustainable-spending constraints. We assume that current-use gifts are a fraction g_u of wealth w_t, while endowment gifts are a fraction g_e. Then the intertemporal budget constraint becomes

$$dw_t = w_t dr_{p,t} + w_t(g_u + g_e)dt - c_t dt, \tag{50}$$

and the arithmetic sustainable-spending constraint becomes

$$c_t dt = w_t \left(\mathrm{E}_t dr_{p,t} + g_u \right) = w_t(r_f + g_u + \alpha\mu)dt. \tag{51}$$

Both types of gifts enter equation (50), but only current-use gifts enter equation (51). Similarly, only current-use gifts enter the geometric sustainable-spending constraint.

It is straightforward to show that current use gifts are equivalent to an increase in the riskfree interest rate, and therefore they discourage risktaking. Endowment gifts, on the other hand, are equivalent to an increase in the rate of time preference, and therefore they encourage risktaking. These results hold whether the sustainable spending constraint is arithmetic or geometric.

3.3 A Nominal Spending Constraint with Inflation

So far we have considered a real model where consumption, asset returns, and sustainable spending constraints are measured in real terms. We now extend our analysis to consider a nominal spending constraint in a model with inflation.

Consider a price level p_t that follows

$$dp_t = p_t \pi dt, \tag{52}$$

where π is the constant inflation rate. The nominal rate becomes

$$r_f^{\$} = r_f + \pi \tag{53}$$

and the nominal return on the risky asset is

$$dr_t^{\$} = (r_f + \pi + \mu)dt + \sigma dZ_t. \tag{54}$$

Arithmetic Average Model Suppose that the investor has a nominal sustainable spending constraint

$$c_t^{\$} dt = w_t^{\$} \mathrm{E}[dr_{p,t}^{\$}], \tag{55}$$

where $c_t^\$ = c_t p_t$ and $w_t^\$ = w_t p_t$, so that

$$c_t dt = w_t \mathrm{E}[dr_{p,t}^\$] = w_t(r_f + \pi + \alpha\mu)dt. \tag{56}$$

Nominal wealth then follows a martingale, while real wealth follows the process

$$\frac{dw_t}{w_t} = \frac{dw_t^\$}{w_t^\$} - \pi dt = -\pi dt + \alpha\sigma dZ_t, \tag{57}$$

and log consumption follows

$$d\log(c_t) = \left(-\pi - \frac{\alpha^2\sigma^2}{2}\right) dt + \alpha\sigma dZ_t. \tag{58}$$

We can now rewrite the indifference condition and portfolio constraint as

$$c_0 = \left[\left(\rho - (\gamma-1)\pi - \gamma(\gamma-1)\frac{\sigma_c^2}{2}\right)(1-\gamma)v\right]^{\frac{1}{1-\gamma}} \tag{59}$$

and

$$c_0 = r_f + \pi + \left(\frac{\mu}{\sigma}\right)\sigma_c. \tag{60}$$

Comparing these equations with the real versions (40) and (41), we can see that inflation subtracts $(\gamma-1)\pi$ from the rate of time preference and adds π to the riskfree interest rate. Both effects reduce risktaking. Hence, with a given real interest rate, a higher inflation rate (equivalently, a higher nominal interest rate) reduces risktaking.

Even if we fix the nominal interest rate, lowering r_f to offset any change in π, an increase in inflation is equivalent to a reduction in the rate of time preference and hence it lowers risktaking.

Geometric Average Model Very similar results hold in a model with a nominal geometric average constraint. Here we have

$$c_t dt = w_t \left(r_f + \pi + \alpha\mu - \frac{1}{2}\alpha^2\sigma^2 \right) dt \tag{61}$$

and

$$d\log(c_t) = \left(-\pi - \frac{\alpha^2\sigma^2}{2} \right) dt + \alpha\sigma dZ_t. \tag{62}$$

We can rewrite the indifference condition and portfolio constraint as

$$c_0 = \left[\left(\rho - (\gamma-1)\pi - (\gamma-1)^2 \frac{\sigma_c^2}{2} \right) (1-\gamma)v \right]^{\frac{1}{1-\gamma}} \tag{63}$$

and

$$c_0 = r_f + \pi + \left(\frac{\mu}{\sigma} \right) \sigma_c - \frac{1}{2}\sigma_c^2. \tag{64}$$

Once again, we see that higher inflation is equivalent to a lower rate of time preference and a higher riskfree interest rate, so it lowers risktaking.

These results allow our model to explain why reaching for yield has been a particular concern in recent years, when it was not during the 1970s. In that decade, the riskfree real interest rate was low but inflation was high, so investors with nominal sustainable spending constraints would not have reached for yield.

3.4 Epstein-Zin Preferences

In this section we consider an investor who has Epstein-Zin preferences (Epstein and Zin 1989, 1991), specified in continuous time following Duffie and Epstein (1992). These preferences allow the elasticity of intertemporal substitution (EIS), which we denote by ψ, to differ from the reciprocal of the coefficient of relative risk aversion γ. We show that the properties of the power utility case with $\gamma > 1$, given in Propositions 1 and 2, carry through regardless of the value of the EIS.

Utility for an infinitely lived investor with Epstein-Zin preferences is defined as the solution to

$$V_t = \mathrm{E}_t \int_t^{\infty} f(c_s, V_s) ds, \tag{65}$$

where

$$f(c, V) = \frac{1}{1-\psi^{-1}} \left(\frac{\rho c^{1-\psi^{-1}}}{((1-\gamma)V)^{\frac{\gamma-\psi^{-1}}{1-\gamma}}} - \rho(1-\gamma)V \right). \tag{66}$$

The appendix writes out the HJB equation and shows how to solve the model; here we simply state the results for the standard unconstrained model and our model with arithmetic and geometric sustainable spending constraints.

Unconstrained model In the standard unconstrained model, the portfolio rule is the same as for power utility: $\alpha = \mu/\gamma\sigma^2$. The consumption-wealth ratio is given by

$$\frac{c_t}{w_t} = \frac{\rho}{\psi^{-1}} + \frac{\psi^{-1}-1}{\psi^{-1}} \left(r_f + \frac{1}{2\gamma} \left(\frac{\mu}{\sigma} \right)^2 \right), \tag{67}$$

which coincides with the power utility formula (11) when $\gamma = \psi^{-1}$.

Arithmetic constraint In a model with an arithmetic sustainable spending constraint, the optimal risky share is

$$\alpha = \frac{-r_f + \sqrt{L}}{\mu(1+\psi^{-1})}, \tag{68}$$

where

$$L = r_f^2 + 2\rho \left(\frac{1+\psi^{-1}}{\gamma} \right) \left(\frac{\mu}{\sigma} \right)^2. \tag{69}$$

This has all the same properties given in Proposition 1 for the power utility case. In addition, it has the property that the risky share approaches zero as the EIS ψ approaches zero, regardless of the level of risk aversion.

Geometric constraint In a model with a geometric sustainable spending constraint, the appendix shows that the risky share has all the same properties given in Proposition 2 for the power utility case. In addition, it has the property that the risky share is increasing in the EIS ψ.

4 A Dynamic Model

Earlier sections of this paper have compared static equilibria with higher or lower real interest rates that are assumed to be constant over time. It is natural to ask whether similar results apply in a dynamic model where the riskfree interest rate moves over time.

In a dynamic environment where investment opportunities change over time, the demand for a risky asset depends on its intertemporal hedging properties (Merton 1973). As in Campbell and Viceira (2001), we allow the riskfree interest rate to move but assume a constant risk premium. Then, intertemporal hedging is driven by correlation between shocks to the return on the risky asset and shocks to the riskfree interest rate. We capture such correlation by allowing interest rate innovations to load on two independent Brownian motions, one of which also drives the return on the risky asset. To see how this works, assume two independent Brownian motions dZ_{1t} and dZ_{2t}. Then a third Brownian motion $dZ_{3t} = \eta dZ_{1t} + \sqrt{1-\eta^2} dZ_{2t}$ has instantaneous variance dt and instantaneous correlation ηdt with dZ_{1t}.

The full problem of the agent now becomes

$$\max_{\alpha_t} E_0 \int_0^\infty e^{-\rho t} u(c_t) dt \tag{70}$$

$$\begin{aligned} \text{subject to } c_t &= w_t \left(r_{ft} + \alpha_t \mu - \frac{1}{2}\alpha_t^2 \sigma^2 \right) \\ \begin{pmatrix} dw_t \\ dr_{ft} \end{pmatrix} &= \begin{pmatrix} \frac{1}{2} w_t \alpha_t^2 \sigma^2 \\ \phi(r_{ft}) \end{pmatrix} + \begin{pmatrix} w_t \alpha_t \sigma & 0 \\ \nu r_{ft} \eta & \nu r_{ft} \sqrt{1-\eta^2} \end{pmatrix} \begin{pmatrix} dZ_{1t} \\ dZ_{2t} \end{pmatrix} \end{aligned} \tag{71}$$

HJB equation Under these assumptions the HJB equation is

$$\rho v(w_t, r_{ft}) = \max_{\alpha} \left\{ u(c_t) + \frac{\partial v}{\partial w} \frac{1}{2} w_t \alpha^2 \sigma^2 + \frac{\partial v}{\partial r_f} \phi(r_{ft}) + \frac{1}{2} \frac{\partial^2 v}{\partial w^2} w_t^2 \alpha^2 \sigma^2 + \frac{1}{2} \frac{\partial^2 v}{\partial r_f^2} \nu^2 r_{ft}^2 + \frac{\partial^2 v}{\partial w \partial r_f} w_t \alpha \sigma \nu r_{ft} \eta \right\}. \tag{72}$$

As previously, we conjecture that the value function takes the form

$$v(w_t, r_{ft}) = A(r_{ft}) \frac{w_t^{1-\gamma}}{1-\gamma}, \tag{73}$$

which differs from the static model by having a time-varying coefficient $A(r_{ft})$ that reflects changing investment opportunities. If we substitute this guess for the value function into the HJB equation, wealth cancels thus verifying the conjectured form of the value function. We collect terms to obtain

$$\begin{aligned} \rho A(r_{ft}) \frac{1}{1-\gamma} = \max_{\alpha} \frac{\left(r_{ft} + \alpha\mu - \frac{1}{2}\alpha^2\sigma^2 \right)^{1-\gamma}}{1-\gamma} &+ A(r) \frac{1}{2} (1-\gamma) \alpha^2 \sigma^2 + A'(r_{ft}) \frac{1}{1-\gamma} \phi(r_{ft}) \\ &+ \frac{1}{2} A''(r_{ft}) \frac{1}{1-\gamma} \nu^2 r_{ft}^2 + A'(r_{ft}) \alpha \sigma \nu r_{ft} \eta. \end{aligned} \tag{74}$$

First-order condition The first order condition for this problem has an additional term compared to the static model:

$$\left(r_{ft} + \alpha\mu - \frac{1}{2}\alpha^2\sigma^2\right)^{-\gamma}(\mu - \alpha\sigma^2) = A(r_{ft})(\gamma - 1)\alpha\sigma^2 - A'(r_{ft})\sigma r_{ft}\nu\eta. \tag{75}$$

The first-order condition shows that α depends on the correlation between the unexpected risky asset return and the risk free rate η. A lower correlation η decreases the right-hand side of equation (75) since $A'(r_{ft}) < 0$. The left-hand side is decreasing in α, so the optimal risky share α is larger for a smaller correlation η.

The drift of log consumption In the static model the geometric average sustainable spending rule implies that the logs of both wealth and consumption are martingales. In the dynamic model, log wealth remains a martingale but log consumption is not in general. To see this, consider the log change in consumption

$$\begin{aligned} d\log c_t &= d\log w_t \left(r_{ft} + \alpha(r_{ft})\mu - \frac{1}{2}\alpha(r_{ft})^2\sigma^2\right) \\ &= \underbrace{d\log w_t}_{\text{mean zero}} + d\log\left(r_{ft} + \alpha(r_{ft})\mu - \frac{1}{2}\alpha(r_{ft})^2\sigma^2\right). \end{aligned} \tag{76}$$

In general the second term has nonzero drift and we cannot characterize it analytically. Therefore, we rely on numerical methods to describe this and other properties of the solution. The online appendix provides more details on the numerical implementation.

Analysis of the numerical solution In Figure 9, we present a numerical solution for the dynamic model and illustrate its sensitivity to different parameter values. We set

$$\phi(r_{ft}) = \frac{1}{2}\nu^2 r_{ft}^2 \tag{77}$$

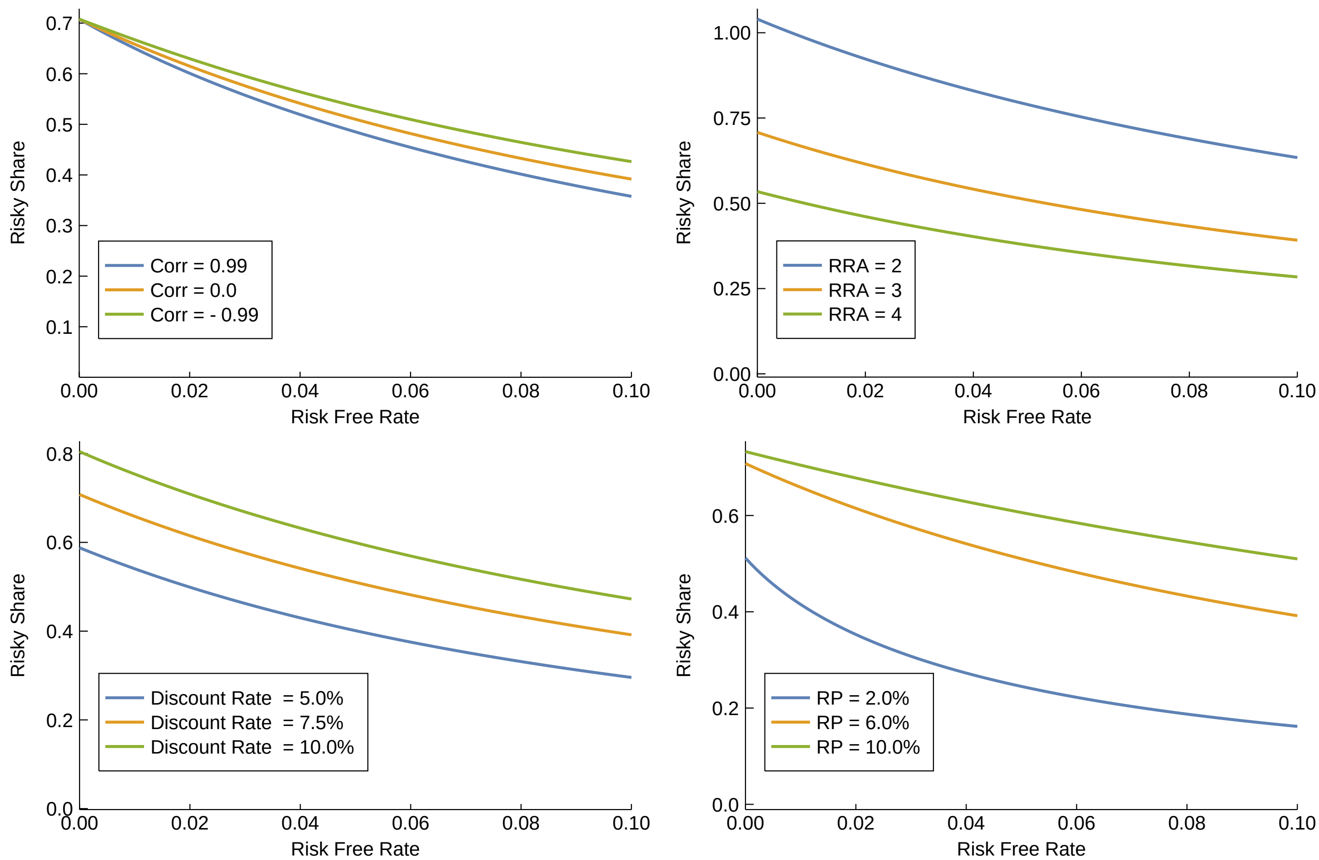

Figure 9: Sensitivity of Optimal Risky Share to Parameters in Dynamic Model

in order to make $\log(r_{ft})$ a random walk. This process is natural for our application since it implies that log consumption for an agent that takes zero risk ($\alpha = 0$) follows a random walk similarly to the static case. Log consumption for the agent who invests optimally is not quite a random walk, but predictable changes in consumption are small. More generally, a persistent process for the riskfree rate is required to avoid transient movements in consumption and asset allocation that would be generated by a sustainable spending constraint interacting with transitory shocks to the expected return.

All four panels of Figure 9 have the riskfree interest rate on the horizontal axis, and the risky share on the vertical axis. All the panels show decreasing convex curves, consistent with the static analysis of reaching for yield. The panels differ in the parameters assumed. In every panel the base case, shown as a middle orange curve, has a zero correlation between the riskfree rate and the risky asset return, risk aversion of 3, a time discount rate of 7.5%, a risk premium of 6%, and a risky standard deviation of 18% implying a Sharpe ratio of 1/3 and Merton risky share of 0.62. We set the volatility of risk free rate $\nu = 1\%$.

The top left panel varies the correlation to 0.99 or -0.99. A negative correlation increases the demand for the risky asset while a positive correlation reduces it, consistent with the logic of Merton (1973, 1993) and Campbell and Viceira (2001). Quantitatively, we find that this hedging demand is very similar in magnitude to that in the standard Merton model for all values of the riskfree interest rate. The top right panel varies risk aversion up to 4 or down to 2, with intuitive effects on risktaking. The bottom left panel varies the rate of time preference up to 10% or down to 5%, and as in the static model a more impatient investor takes more risk. The bottom right panel varies the risk premium up to 10% or down to 2%. Since in this model the riskfree interest rate is always positive, the effect of the risk premium on risktaking is always positive, but the magnitude of the effect is smaller when the interest rate reaches very low levels. These findings confirm the main results of our static analysis for a plausible dynamic model.

5 Conclusion

In this paper we have shown that a constraint on an investor's ability to save or dissave can break the standard result that risktaking depends only on risk aversion, risk, and the risk premium available in financial markets. An investor with a sustainable spending constraint reaches for yield, taking more risk as the riskfree interest rate declines, even if all the standard determinants of risktaking are constant. Furthermore, the tendency to reach for yield is stronger when the real interest rate is low than when it is high. This may be one reason why reaching for yield has been so widely discussed in the low-interest-rate environment of the early 21st Century.

Reaching for yield also changes the investor's response to a change in the risk premium. An increase in the risk premium stimulates risktaking through the conventional channel, a substitution effect towards risky investing. However it also weakens reaching for yield, an offsetting income effect that reduces risktaking. The offsetting effect becomes stronger when the riskfree interest rate is low, and can even dominate at sufficiently low levels of the interest rate.

For any given preference parameters, there is a level of the riskfree interest rate at which a sustainable spending constraint does not bind because the unconstrained investor does not wish to accumulate or decumulate wealth. The welfare cost of a sustainable spending constraint is very low when the riskfree rate is near this level, but increases rapidly as the riskfree rate declines and more slowly as the riskfree rate increases. A one-sided constraint, preventing wealth decumulation but allowing wealth accumulation, binds only for rates below this level; this provides another reason why reaching for yield may be more important today than in earlier decades where interest rates were higher.

We have extended our model to consider what happens when the investor receives a stream of gifts, finding that current-use gifts moderate risktaking while endowment gifts increase it. We have also considered an alternative model in which the sustainable spending constraint is specified in nominal terms. In this case low inflation stimulates risktaking even at a constant real interest rate; this may help to explain why reaching for yield was less of a concern during the 1970s, when the real interest rate was low but inflation was high.

Most of our analysis is static, comparing equilibria with permanently different levels of the riskfree interest rate. However we show that our results carry over to the numerical solution of a dynamic model with a highly persistent riskfree rate whose log follows a random walk.

Although our results depend on the functional forms we have assumed—both the power form for utility and the arithmetic or geometric forms for the sustainable spending constraints—we believe the insights of this paper are more general. Most obviously, the paper shows that the analysis extends straightforwardly to the case of Epstein-Zin preferences. It should also be possible to enrich our model to allow for a flexible constraint that penalizes but does not prohibit saving or dissaving. The general lesson is that the classic separation between the riskfree interest rate and risktaking is critically dependent on the assumption that investors can freely adjust their spending plans.

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